DISTRIBUTIONS DERIVED FROM THE MULTINOMIAL EXPANSION

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1. Introduction. This exposition describes certain probability distributions associated with the multinomial expansion. It begins with a description of the graph of the multinomial distribution. Then the following three topics are treated: (a) the probability distribution for the number of classes in a sample which contain at least one variate; (b) an elementary test for goodness of fit and remarks on the Chi-Square Test; (c) the distribution of $(y+\cdots+y)$ in terms of the distribution of y.

The binomial distribution is an important feature of a first course in statistics, but distributions associated with the multinomial expansion are not emphasized to the same extent. Many text books give a derivation of the normal probability surface from the multinomial expansion but go no further. Others refer to the "multinomial distribution, the forms of which are so many and so various as almost to defeat classification," but include only one or two simple examples. It is hoped that the three problems treated here, each of importance in itself, will give some indication of the rôle of the multinomial distribution.

2. The multinomial distribution. Consider an event characterized by a random variable y that takes on one and only one of the values $1, 2, \cdots, k$ on each trial. Let the probability that y takes the value i on a single trial of the event be p_i. Then $p_1+p_2+\cdots+p_k=1$. From well known laws of the subject, the probability that y has the value 1 f_1 times, that it has the value 2 f_2 times, $\cdots$, and that it has the value k f_k times, in some order, on n trials is

$$M(f_1, f_2, \cdots, f_k) = \frac{n!}{f_1!f_2!\cdots f_k!} p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}, \tag{2.1}$$

where $f_1+f_2+\cdots+f_k=n$. All possible outcomes correspond to all possible terms of the form (2.1), and they are the terms in the expansion of $(p_1+p_2+\cdots+p_k)^n$. The probability distribution which consists of all the terms (2.1) is called the multinomial distribution* for obvious reasons (see [1], p. 50 and [6], pp. 58–59). The special case in which $k=2$ is the well-known binomial distribution.

It will be necessary for later purposes to have a clear understanding of the nature of the graph of the multinomial distribution. The special case $k=3$ will be examined first because it is typical, and in addition the essential features of it can be drawn explicitly and visualized in ordinary 3-space.

The general term in $(p_1+p_2+p_3)^n$ is

$$M(f_1, f_2, f_3) = \frac{n!}{f_1!f_2!f_3!} p_1^{f_1} p_2^{f_2} p_3^{f_3}, \tag{2.2}$$

* Standard notation would require the use of $x_1, x_2, \cdots, x_k$ in place of $f_1, f_2, \cdots, f_k$, but it is felt that the use of the f's, suggesting frequencies, will contribute to the ease of reading certain parts of this paper.

and it is associated with the point (f_1, f_2, f_3) in euclidean 3-space. The value of the term can be plotted as an ordinate in the fourth dimension of an (f_1, f_2, f_3, M) space, but for the purposes of this paper it will be better to consider that the term (2.2) is merely marked at the point (f_1, f_2, f_3). Then, since $f_1+f_2+f_3=n$, the terms of the multinomial distribution are marked at the points which have integral coordinates and which lie on the equilateral triangle cut out of the plane $f_1+f_2+f_3=n$ by the first octant. This multinomial distribution has the following important property: the terms marked along any line $f_1=$constant of the triangle are, except for a constant factor, the terms of a binomial distribution with respect to f_2 and f_3; similarly for terms along the lines $f_2=$constant and $f_3=$constant. In particular, the terms marked along the lines $f_1=0$, $f_2=0$, and $f_3=0$ are exactly the terms of the binomial distributions $(p_2+p_3)^n$, $(p_3+p_1)^n$, $(p_1+p_2)^n$.

In the general case, the term $M(f_1, f_2, \cdots, f_k)$ is marked at the point $(f_1, f_2, \cdots, f_k)$ in k-dimensional euclidean space. The terms $M(f_1, f_2, \cdots, f_k)$ are thus marked at the lattice points of a $(k-1)$-cell in k-space. The property stated in the special case and generalizations of it hold for the general multinomial distribution $M(f_1, f_2, \cdots, f_k)$.

The Number of Classes in a Sample which Contain at Least one Variate

3. Statement of the problem. Let the random variable y of section 2 have the value i if and only if a variate drawn at random from a certain population lies in the ith class interval of the frequency distribution. Then p_i denotes the probability that a variate drawn at random will lie in the ith class interval. If a sample of n variates is drawn, there is a certain probability $P(x; n, k)$ that the n variates lie in exactly x, $1 \leqq x \leqq k$, of the k class intervals. The problem is the following: to determine the probability distribution of x and also its properties.

4. Determination of the distribution of x in the most general case. It will be instructive to consider the special case $k=3$ before the solution of the general problem is undertaken. The term $M(f_1, f_2, f_3)$ in (2.2) is the probability that a sample of n variates will be drawn in which the classes 1, 2, 3 have frequencies f_1, f_2, f_3 respectively. The probability of $x=1$ is the probability of drawing a sample in which two of the three frequencies f_1, f_2, f_3 are zero and the third is n. An examination of the graph of $M(f_1, f_2, f_3)$, described in section 2 above, shows that the terms for which two of the three frequencies are zero are exactly those terms which are marked at the three vertices of the triangle. Thus $P(1; n, 3)$ is the sum of these three terms. The probability of $x=2$ is the probability of drawing a sample in which exactly one class frequency is zero; the corresponding terms $M(f_1, f_2, f_3)$ are those marked along the sides, but not at the vertices, of the triangle. Thus $P(2; n, 3)$ is the sum of the terms marked on the sides but not at the vertices of the triangle. Finally, each term plotted on the interior of the triangle gives the probability of some sample for which $x=3$, and $P(3; n, 3)$ is the sum of all these terms. These facts, together with the fact stated in section 2 that the binomial distributions $(p_2+p_3)^n$, $(p_3+p_1)^n$ and $(p_1+p_2)^n$ are plotted along the sides of the triangle, yield the formulas for $P(x; n, 3)$. They are:

$$(4.1)\quad \begin{cases} P(1; n, 3) = & (p_1^n + p_2^n + p_3^n), \\ P(2; n, 3) = & [(p_2+p_3)^n + (p_3+p_1)^n + (p_1+p_2)^n] - 2(p_1^n + p_2^n + p_3^n), \\ P(3; n, 3) = (p_1+p_2+p_3)^n - & [(p_2+p_3)^n + (p_3+p_1)^n + (p_1+p_2)^n] + (p_1^n + p_2^n + p_3^n). \end{cases}$$

Consider the expression for $P(2; n, 3)$. The term $[(p_2+p_3)^n+(p_3+p_1)^n+(p_1+p_2)^n]$ contains all terms plotted along the sides, including those at the vertices, of the triangle. Since each vertex belongs to two sides of the triangle, it is necessary to subtract twice the sum $(p_1^n+p_2^n+p_3^n)$ to obtain $P(2; n, 3)$. Similarly, in computing $P(3; n, 3)$, $(p_1+p_2+p_3)^n$ gives all the terms in the triangle. It is necessary to subtract $[(p_2+p_3)^n+(p_3+p_1)^n+(p_1+p_2)^n]$ in order to remove the terms on the sides; but since each vertex belongs to two sides, this step removes the terms at the vertices twice. It is thus necessary to add $(p_1^n+p_2^n+p_3^n)$ in order to obtain $P(3; n, 3)$.

The method which has been applied in the special case can be extended to the general case. Let s_j denote the sum of j of the probabilities $p_1, p_2, \cdots, p_k$, and let $S_j=\sum s_j^n$, where the summation extends over all possible combinations of j probabilities that can be selected from $p_1, p_2, \cdots, p_k$. Arguments similar to those employed in the special case lead to the following result, where the C's denote the binomial coefficients as usual:

$$(4.2)\quad \begin{cases} P(1; n, k) = & {}_{k-1}C_0\ S_1, \\ P(2; n, k) = & {}_{k-2}C_0 S_2 + (-1)\ {}_{k-1}C_1\ S_1, \\ \cdots\cdots\cdots\

Let the mean and standard deviation of the distribution $P(x; n, k)$, for fixed n and k, be denoted by $\bar{x}_k$ and σ_k respectively. From (5.1) we have

$$(k - x)P(x; n, k) = k(1 - 1/k)^n P(x; n, k - 1), \tag{5.4}$$

$$x(k - x)P(x; n, k) = xk(1 - 1/k)^n P(x; n, k - 1). \tag{5.5}$$

Summation of these equations with respect to x, using (5.3), gives

$$k - \bar{x}_k = k(1 - 1/k)^n,$$

$$k\bar{x}_k - (\bar{x}_k^2 + \sigma_k^2) = k(1 - 1/k)^n \bar{x}_{k-1} = (k - \bar{x}_k)\bar{x}_{k-1}.$$

Solution of these equations for $\bar{x}_k$ and σ_k^2 gives

$$\bar{x}_k = k[1 - (1 - 1/k)^n], \tag{5.6}$$

$$\sigma_k^2 = (k - \bar{x}_k)(\bar{x}_k - \bar{x}_{k-1}). \tag{5.7}$$

It follows that

$$\lim_{n\to\infty} \bar{x}_k = k; \qquad \lim_{n\to\infty} \sigma_k^2 = 0; \tag{5.8}$$

$$\bar{x}_k = \bar{x}_{k-1} = 1, \quad n = 1, \qquad \bar{x}_k > \bar{x}_{k-1}, \quad n \geqq 2. \tag{5.9}$$

Furthermore,

$$\lim_{n\to\infty} P(x; n, k) = 0, \quad 1 \leqq x \leqq k - 1, \qquad = 1, \quad x = k. \tag{5.10}$$

6. A second derivation and a recurrence relation. A second derivation can be given for the probability distribution (5.1), and it will be included here to bring out an important recurrence relation. First, it will be necessary to recall the following properties of the "differences of zero":

$$\Delta^x 0^n = 0, \quad x > n; \qquad \Delta^n 0^n = n!; \tag{6.1}$$

$$\Delta^x 0^{n+1} = x\Delta^{x-1}0^n + x\Delta^x 0^n. \tag{6.2}$$

Proofs of these relations will be found in [3], p. 36.

Consider now the determination of $P(x; n, k)$ in the special case $p_i = 1/k$, $i = 1, 2, \cdots, k$. It is clear that $P(1; 1, k) = 1$, and that $P(x; 1, k) = 0$ for $x > 1$ since only one class can contain a variate if only one is drawn. Consider next the situation when $n = 2$. Now $P(1; 2, k) = 1/k$, for the two variates belong to the same class only under the following conditions: the first variate belongs to an arbitrary class, but the second variate belongs to the same class as the first one. The probability of the first event is 1, and that of the second is $1/k$; the probability that both events happen is thus their product $1/k$. Next, $P(2; 2, k) = (k-1)/k$ by the same argument, for the two variates belong to different classes only if the first belongs to one class and the second to some one of the remaining $(k-1)$ classes. As before, $P(x; 2, k) = 0$ for $x > 2$.

Suppose $P(x; n-1, k)$ has been determined for $x=1, 2, \cdots, k$. It will be shown that the probability distribution $P(x; n, k)$ also can be determined. The n variates of a sample occupy exactly x classes if and only if one of the following two situations occurs: (a) the first $(n-1)$ variates drawn occupy $(x-1)$ classes and the last variate drawn falls in one of the remaining $k-(x-1)$ classes; (b) the first $(n-1)$ variates occupy x classes, and the last variate falls in one of these same x classes. It follows that

$$(6.3) \quad P(x; n, k) = \frac{k-x+1}{k} P(x-1; n-1, k) + \frac{x}{k} P(x; n-1, k),$$

where it is convenient to define $P(x-1; n-1, k)=0$ for $x=1$. A complete induction can now be carried through to show that the probability distribution $P(x; n, k)$, $x=1, 2, \cdots, k$, can be calculated for every value of n.

Finally, it can be shown that the distribution is given by the formula (5.1). First, it can be shown from (5.2) and (6.1) that the formula (5.1) yields the values that have been calculated for $P(x; 1, k)$ and $P(x; 2, k)$. Assume next that the formula (5.1) holds for the distribution $P(x; n-1, k)$. Substitute from (5.1) in (6.3) and simplify by means of (6.2); it follows that the formula (5.1) holds also for the distribution $P(x; n, k)$. A complete induction then establishes the validity of (5.1) for all x, n, and k.

The second derivation of the probability distribution $P(x; n, k)$, based on the fundamental recurrence relation (6.3), is thus complete.

It may be observed that (5.4) is satisfied by the distributions $P(x; n, k)$.

The nature of the distribution and also the variation of the distribution as n increases can be observed by plotting the distributions for $n=1, 2, \cdots, 15$ from the values given in Table I.

TABLE I. Values of $P(x; n, 10)$ for $n=1, 2, \cdots, 15$
$p_i=1/10$, $i=1, 2, \cdots, 10$

n	x									
	1	2	3	4	5	6	7	8	9	10
1	1.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2	.1000	.9000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
3	.0100	.2700	.7200	.0000	.0000	.0000	.0000	.0000	.0000	.0000
4	.0010	.0630	.4320	.5040	.0000	.0000	.0000	.0000	.0000	.0000
5	.0001	.0135	.1800	.5040	.3024	.0000	.0000	.0000	.0000	.0000
6	.0000	.0028	.0648	.3276	.4536	.1512	.0000	.0000	.0000	.0000
7	.0000	.0006	.0217	.1764	.4234	.3175	.0605	.0000	.0000	.0000
8	.0000	.0001	.0070	.0857	.3175	.4022	.1693	.0181	.0000	.0000
9	.0000	.0000	.0022	.0392	.2102	.4001	.2794	.0653	.0036	.0000
10	.0000	.0000	.0007	.0172	.1286	.3451	.3556	.1361	.0163	.0004
11	.0000	.0000	.0002	.0073	.0746	.2714	.3870	.2156	.0419	.0020
12	.0000	.0000	.0001	.0031	.0417	.2001	.3794	.2885	.0808	.0062
13	.0000	.0000	.0000	.0013	.0227	.1409	.3457	.3447	.1305	.0143
14	.0000	.0000	.0000	.0005	.0121	.0959	.2983	.3794	.1863	.0273
15	.0000	.0000	.0000	.0002	.0064	.0636	.2472	.3930	.2436	.0460

7. Relation of sample size to the number of class intervals occupied. In certain applications it is necessary to know how many variates must be distributed in the k classes in order that, on the average, a certain proportion of them will contain at least one variate. The answer is obtained from (5.6). Set $\bar{x}=tk$, $0<t<1$. Then

$$n = \frac{\log (1-t)}{\log (1-1/k)}. \tag{7.1}$$

If k is large, then a first approximation gives

$$n \sim -k \log (1-t). \tag{7.2}$$

Thus, when k is large, the number of variates that must be distributed among the k classes so that, on the average, a certain proportion t of the classes shall contain at least one variate is approximately proportional to k. This result and the values in Table II are important in certain of the applications.

TABLE II. The Approximate Value of n for which $\bar{x}=tk$

t	.25	.50	.75	.90	.95	.99
n	$.29k$	$.69k$	$1.39k$	$2.30k$	$3.00k$	$4.61k$

An Elementary Test for Goodness of Fit and Remarks on the Chi-Square Test

8. Statement of the problem and remarks. Consider again the situation described in section 3 above: a sample of n variates is drawn under conditions such that the probability that each variate will belong to the ith class interval of the sample is p_i, $i=1, 2, \cdots, k$. The problem is to obtain a criterion in terms of which it can be determined whether the sample actually drawn is a likely or unlikely one on the basis of chance. It will be in order to make this very general statement of the problem more precise.

The probability that a sample with class frequencies $f_1, f_2, \cdots, f_k$ will be drawn is given by the expression $M(f_1, f_2, \cdots, f_k)$ in (2.1), but this information in itself does not solve the problem. The solution must involve in some form a comparison of the probabilities of all possible samples. The first step will be to determine the most likely sample and in general to order all samples in terms of the magnitudes of their probabilities. Then when a sample S is drawn, the following procedure will give an estimate of whether it is likely or unlikely. Divide the class of all samples into two classes A and B as follows: in class A place all samples with a greater probability than the given sample S, and in B place all samples with the same or a smaller probability than S. A comparison of the sum of the probabilities of all samples in class A with the sum of the probabilities of all samples in class B is then the criterion for determining whether the sample S is likely on the basis of chance. If the former sum is large in comparison with the latter (the definition of "large" is the specification of the significance level), we say the sample S is unlikely on the basis of chance.

One final remark is in order. The steps outlined involve only simple arithmetic, and they can be carried out without difficulty in case both the number of classes k and the size n of the sample S are small. It is at this point that the Chi-Square Test comes in: it supplies a means for approximating the classes A and B and the probabilities in question with ease, sufficient accuracy, and speed so that the test becomes a workable one in the applications.

9. The most probable sample. The class frequencies of that sample of n variates which is most likely, that is, which has a greater probability than any other single sample, are the exponents of the largest term in the expansion of $(p_1+p_2+\cdots+p_k)^n$. In certain cases at least it is possible to predict the result entirely on the basis of probability considerations. It is to be expected that, when $p_1, p_2, \cdots, p_k$ and n have values such that $np_1, np_2, \cdots, np_k$ are integers, the most likely sample is the one for which $f_1=np_1, f_2=np_2, \cdots, f_k=np_k$. The result is strictly an arithmetic one and an arithmetic proof can be given. Let

$$\frac{n!}{m_1!m_2!\cdots m_k!}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k} \tag{9.1}$$

be the maximum term in the expansion of $(p_1+p_2+\cdots+p_k)^n$ or one of the maximum terms in case there are several. Consider the terms

$$\frac{n!}{f_1!f_2!m_3!\cdots m_k!}p_1^{f_1}p_2^{f_2}p_3^{m_3}\cdots p_k^{m_k}. \tag{9.2}$$

They are the terms of a binomial expansion multiplied by a constant, and they will have a maximum for $f_1=m_1$, $f_2=m_2$. Since the term (9.1) is equal to or greater than the two terms adjacent to it in the expansion (9.2), it follows that m_1 and m_2 are solutions of the inequalities $p_2(f_1+1)\geqq p_1f_2$, $p_1(f_2+1)\geqq p_2f_1$. Similar considerations apply to each pair of exponents. Thus a necessary condition that (9.1) be a maximum term is that $f_1=m_1, \cdots, f_k=m_k$ be a solution of

$$\begin{cases} p_j(f_i+1)\geqq p_if_j, \\ p_i(f_j+1)\geqq p_jf_i, \end{cases} \tag{9.3}$$

where i, j take on each distinct pair of values in $1, 2, \cdots, k$.

Now suppose that $np_1, np_2, \cdots, np_k$ are integers. Since their sum is n, they are the exponents of one of the terms in the expansion. Furthermore, $f_i=np_i$, $f_j=np_j$ is a solution of the inequalities (9.3). Finally, the necessary condition is also sufficient in this case. Let the term

$$\frac{n!}{(np_1)!(np_2)!\cdots(np_k)!}p_1^{np_1}p_2^{np_2}\cdots p_k^{np_k} \tag{9.4}$$

be divided by any other term in the expansion $(p_1+p_2+\cdots+p_k)^n$, the latter term being conveniently written in the form

$$\frac{n!}{(np_1+t_1)!(np_2+t_2)!\cdots(np_k+t_k)!}p_1^{np_1+t_1}p_2^{np_2+t_2}\cdots p_k^{np_k+t_k} \tag{9.5}$$

where $t_1+t_2+\cdots+t_k=0$. Inspection shows that a quotient greater than 1 is obtained. The proof is thus complete that (9.4) is the maximum term, and that $np_1, np_2, \cdots, np_k$ are the class frequencies of the most likely sample.

It should be observed that the term (9.4) is the term which occurs at the mean of the multinomial distribution $M(f_1, f_2, \cdots, f_k)$ (see section 2 above).

The situation in case $np_1, np_2, \cdots, np_k$ are not all integers is not easily described. The maximum term still occurs near the mean of the distribution, but there may be several equal terms. We shall not explore these cases further.

10. An elementary test of goodness of fit. The remainder of the discussion will be restricted to the special case in which $p_i=1/k$ for $i=1, 2, \cdots, k$. It will be instructive to consider first the case $k=3$, $n=6$. The numbers are small enough in this case so that the probability of each possible sample can be calculated. The results are summarized in Table III. The first column gives all the

TABLE III. Probabilities of Samples and the Order of their Likelihood in the Case $k=3$, $n=6$

Frequencies f_1, f_2, f_3	Probability* of Drawing the Sample f_1, f_2, f_3	Probability* of Drawing a Sample with Frequencies f_1, f_2, f_3 in Some Order	Order of Likelihood	$(f_1-2)^2+(f_2-2)^2+(f_3-2)^2$
2, 2, 2	90	90	1	0
3, 2, 1	60	360	2	2
4, 1, 1	30	90	3	6
3, 3, 0	20	60	4	6
4, 2, 0	15	90	5	8
5, 1, 0	6	36	6	14
6, 0, 0	1	3	7	24

* Each entry in this column is to be divided by $3^6=729$.

distinguishable partitions, apart from order, of the integer 6 into three positive or zero integers. The sets of numbers in this column, together with all of their distinguishable permutations, represent all of the possible samples. The second column gives the probability that a sample will be drawn in which the frequencies in the class intervals 1, 2, 3 are the numbers f_1, f_2, f_3 respectively in the first column. The third column is obtained by multiplying the second column by the number of samples which have the frequencies f_1, f_2, f_3 in some order, that is, by the number of distinguishable permutations of the integers f_1, f_2, f_3. The fourth column gives the order of likelihood of the samples with frequencies f_1, f_2, f_3, namely, the order in the sense of decreasing magnitude of the numbers in the second column. Finally, the last column gives what in the general case would be $\sum_1^k (f_i-n/k)^2$.

It will be observed that the sample with the greatest probability is the one with class frequencies $np_i=n/k=2$ as shown in the last section.

Next, one's attention is caught by the striking fact that the numbers in the last column can almost be used to obtain the order of likelihood of the samples. This result is much less surprising if a plot of this multinomial expansion of the kind described in section 2 is examined. Consider an expanding sphere with center at the mean (and maximum) of the distribution, and recall that along each line f_i = constant of the triangle the distribution is a symmetrical binomial distribution multiplied by a constant. If the expanding sphere is followed as it passes through the successive lattice points of the grid, each new lattice point included being examined in relation to the last ones included, it will be observed that as $(f_1-2)^2+(f_2-2)^2+(f_3-2)^2$ increases, the probability of the sample f_1, f_2, f_3 decreases. It is nevertheless true that in one case there are samples with different probabilities in spite of the fact that they all have the same value of $(f_1-2)^2+(f_2-2)^2+(f_3-2)^2$.

Unfortunately the relations described in the last paragraph do not hold in general. The case of the binomial distribution, however, is altogether special, for a sample f_1, f_2 is more likely than a sample f_1', f_2' if and only if $(f_1-n/2)^2+(f_2-n/2)^2<(f_1'-n/2)^2+(f_2'-n/2)^2$ for every n. For $k=3$, however, the situation changes even for $n=12$. Table IV summarizes the results in this case.

TABLE IV. Probabilities of Samples and the Order of their Likelihood in the Case $k=3$, $n=12$

Frequencies f_1, f_2, f_3	Probability* of Drawing the Sample f_1, f_2, f_3	Probability* of Drawing a Sample with Frequencies f_1, f_2, f_3 in Some Order	Order of Likelihood	$(f_1-4)^2+(f_2-4)^2+(f_3-4)^2$
4, 4, 4	34,650	34,650	1	0
5, 4, 3	27,720	166,320	2	2
6, 3, 3	18,480	55,440	3	6
5, 5, 2	16,632	49,896	4	6
6, 4, 2	13,860	83,160	5	8
7, 3, 2	7,920	47,520	6	14
6, 5, 1	5,544	33,264	7	14
7, 4, 1	3,960	23,760	8	18
8, 2, 2	2,970	8,910	9	24
8, 3, 1	1,980	11,880	10	26
6, 6, 0	924	2,772	11	24
7, 5, 0	792	4,752	12	26
9, 2, 1	660	3,960	13	38
8, 4, 0	495	2,970	14	32
9, 3, 0	220	1,320	15	42
10, 1, 1	132	396	16	54
10, 2, 0	66	396	17	56
11, 1, 0	12	72	18	74
12, 0, 0	1	3	19	96

* Each entry in this column is to be divided by $3^{12}=531,441$.

It may be emphasized finally that a table such as Table IV provides a solution to the problem stated in section 8. Suppose, that to test the hypothesis $p_i = 1/3$ for $i = 1, 2, 3$, a sample of 12 variates is drawn, and that frequencies of 10, 1, 1 are obtained in the three classes. Reference to Table IV shows that this sample is 16th in the order of likelihood. The sum of the last four numbers in the third column of Table IV is 867. Thus, a sample as unlikely as the one drawn (the occurrence of an event in class B of section 8) will be expected only 867 times out of 531,441 trials. On any of the usual significance levels, the drawing of a sample with frequencies 10, 1, 1 is so unlikely on the basis of chance that the hypothesis $p_i = 1/3$, $i = 1, 2, 3$, would certainly be rejected.

The third and fourth columns of Table IV define a probability distribution. The random variable takes on the values listed in the fourth column, and the probability that it takes on any value is given by the corresponding entry in the third column. This probability distribution has been obtained by combining the terms of the multinomial distribution in a certain way. The probability distribution $P(x; n, k)$ also was obtained from the multinomial distribution, but by combining its terms in a different manner.

It may be observed that the probability of drawing the most likely sample is small. Apply the test in the case in which class A contains only the most likely sample, and class B contains all the rest. Table IV shows that the most likely sample would be drawn only 34,650 times out of 531,441 samples. Furthermore, an application of Stirling's Formula to (9.4) shows that the probability of the most likely sample has the limit zero as n tends to infinity; that is, the probability of drawing a sample in class B tends to 1.

The construction of tables such as Table IV becomes laborious for larger values of n and k, but they can be constructed when required. Their chief usefulness would be in the cases of small n and k for which the Chi-Square Test does not give good results. The labor involved in the computation would be immensely greater in case the probabilities $p_1, p_2, \cdots, p_k$ are not all equal. Many of the problems encountered in testing the goodness of fit of a curve can be reduced, however, to the case of equal probabilities by adjusting the class intervals so that the probabilities become equal.

11. The Chi-Square Test. After everything has been said, however, it must be emphasized that it would be impractical to construct any set of tables such as Tables III and IV that would serve for all purposes. Fortunately, however, an approximate solution of the problem was found long ago that is adequate for most purposes. Tables which give the distribution of a quantity known as Chi-Square supply the answer.

Briefly, the Chi-Square solution is the following. A quantity χ^2 is defined by

$$\sum_{i=1}^{k} \frac{(f_i - np_i)^2}{np_i} = \chi^2. \tag{11.1}$$

It should be observed that χ^2 is proportional to the quantity tabulated in the

last column of Tables III and IV when $p_1=p_2=\cdots=p_k=1/k$. Next, the multinomial distribution is approximated by a continuous probability distribution. The procedure in testing whether a given sample is likely on the basis of chance is the following: The value of χ^2 for the given sample is determined from (11.1). The totality of all possible samples is divided into two classes A' and B'; those samples with a value of χ^2 smaller than the given sample are placed in class A', and the remainder are placed in class B'. Reference to the tables of χ^2 then gives an approximate value for the probability that a sample chosen at random will fall in class A' or B'.

It will be observed that the Chi-Square Test approximates the solution which was given in section 10 to the problem stated in section 8. The first difference arises from the fact that the classes A' and B' in the Chi-Square Test are not the same as the classes A and B defined in section 8. The results in section 10, however, indicate that the classes A', B' differ but slightly from the classes A, B. Further approximations arise from replacing the discrete multinomial distribution by a continuous distribution. A full discussion of the latter approximations will be found in [4].

The Distribution of $(y+\cdots+y)$ in Terms of the Distribution of y

12. Statement of the problem. Consider again a random variable y (see section 2), but assume this time that it takes on the values $0, 1, 2, \cdots, k$ with probabilities $p_0, p_1, \cdots, p_k$. The problem is to determine the distribution of the sum of the values y for n variates drawn at random, and also the properties of this distribution.

Consider the probability of drawing n variates for which the sum of the y's is x. The term

$$M(f_0, f_1, \cdots, f_k) = \frac{n!}{f_0!f_1!\cdots f_k!} p_0^{f_0} p_1^{f_1} \cdots p_k^{f_k}, \tag{12.1}$$

in which $f_0+f_1+\cdots+f_k=n$ and $0f_0+1f_1+\cdots+kf_k=x$, is the probability of one of the ways in which a sum x can be obtained. The term $M(f_0, f_1, \cdots, f_k)$ is a term in the multinomial expansion $(p_0+p_1+\cdots+p_k)^n$. Furthermore, the total probability that the n variates drawn will have the sum of the y values equal to x is the sum of all the multinomial terms (12.1) for which $0f_0+1f_1+\cdots+kf_k=x$. It is well known that this probability is exactly the coefficient of t^x in the expansion of $(p_0+p_1t+p_2t^2+\cdots+p_kt^k)^n$, the function $p_0+p_1t+\cdots+p_kt^k$ being the generating function of probability (see [5], pp. 16–20, 55–56 and [6], p. 60).

The problem thus reduces to the determination and study of the coefficients in $(p_0+p_1t+\cdots+p_kt^k)^n$ when this function is expressed as a polynomial in t. It is desirable to generalize the problem as follows: determine the coefficients in $(a_0+a_1t+\cdots+a_kt^k)^n$ and $(a_0+a_1t+\cdots+a_kt^k+\cdots)^n$ when these functions are expressed as a polynomial in t and a power series in t respectively. There are no restrictions on the coefficients $a_0, a_1, \cdots, a_k, \cdots$.

13. Determination of the coefficients. Set

$$(a_0 + a_1 t + \cdots)^n = \sum_{x=1}^{\infty} C(x; n; a_0, a_1, \cdots) t^x. \tag{13.1}$$

The first of three methods that can be applied for the determination and study of $C(x; n; a_0, a_1, \cdots)$ is based on combinatorial considerations. The coefficient $C(x; n; a_0, a_1, \cdots)$ is the sum of all products formed by taking n symbols from the set $a_0, a_1, \cdots$ in all possible orders, repetitions being allowed, but such that the sum of the subscripts equals x. The coefficient of $a_0^{n_0} a_1^{n_1} a_2^{n_2} \cdots$ is the multinomial coefficient

$$\binom{n}{n_1, n_2, \cdots} = \frac{n!}{n_0! n_1! n_2! \cdots}, \qquad n_0 + n_1 + n_2 + \cdots = n. \tag{13.2}$$

Then

$$C(x; n; a_0, a_1, \cdots) = \sum_{s=x} \binom{n}{n_1, n_2, \cdots} a_0^{n_0} a_1^{n_1} a_2^{n_2} \cdots, \tag{13.3}$$

where the summation extends over all values of $n_0, n_1, n_2, \cdots$ such that $s \equiv 0n_0 + 1n_1 + 2n_2 + \cdots = x$ and $n_0 + n_1 + n_2 + \cdots = n$. For example

$$\begin{aligned} C(4; n; a_0, a_1, \cdots) = {} & \binom{n}{4} a_0^{n-4} a_1^4 + \binom{n}{2, 1} a_0^{n-3} a_1^2 a_2 \\ & + \binom{n}{1, 1} a_0^{n-2} a_1 a_3 + \binom{n}{2} a_0^{n-2} a_2^2 \\ & + \binom{n}{1} a_0^{n-1} a_4. \end{aligned} \tag{13.4}$$

The second method that can be used to determine $C(x; n; a_0, a_1, \cdots)$ is based on the theory of functions. If $a_0 + a_1 t + \cdots$ represents an analytic function $f(t)$, then

$$C(x; n; a_0, a_1, \cdots) = \frac{1}{x!} \frac{d^x [f(t)]^n}{dt^x}\bigg|_{t=0}. \tag{13.5}$$

This result is true formally in any case. In many cases this method of determining $C(x; n; a_0, a_1, \cdots)$ is very powerful. For example, if y is distributed according to the Poisson Distribution $e^{-m}[1 + m + m^2/2! + \cdots]$, then an application of (13.5) to the generating function $e^{-m}[1 + mt + m^2 t^2/2! + \cdots] = e^{-m+mt}$ shows that $y + \cdots + y$ (n summands) is distributed according to the Poisson Distribution $e^{-mn}[1 + mn + m^2 n^2/2! + \cdots]$. A similar application of (13.5) to $1 + t + t^2 + \cdots = 1/(1-t)$ shows that

$$C(x; n; 1_0, 1_1, \cdots) = \frac{n(n+1)\cdots(n+x-1)}{x!}. \tag{13.6}$$

The third method for the determination and study of the coefficients $C(x; n; a_0, a_1, \cdots)$ is based on recurrence relations and difference equations. Since $(a_0+a_1t+\cdots)_n=[\sum_1^\infty C(x; n-1; a_0, a_1, \cdots)t^x](a_0+a_1t+\cdots)$, then

$$C(x; n; a_0, a_1, \cdots) = \sum_{i=0}^{x} a_i C(x-i; n-1; a_0, a_1, \cdots). \tag{13.7}$$

This relation is the generalization of the relation that holds in the Pascal Triangle. A generalized Pascal Triangle can be constructed step-wise; the rows give the coefficients in $(a_0+a_1t+\cdots)^n$ successively for $n=1, 2, \cdots$:

$$\begin{cases} C(0; 1; a_0, a_1, \cdots) & C(1; 1; a_0, a_1, \cdots) & \cdots \\ C(0; 2; a_0, a_1, \cdots) & C(1; 2; a_0, a_1, \cdots) & \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \end{cases} \tag{13.8}$$

From another point of view, (13.7) is a set of difference equations by means of which the terms in the nth row of the triangle can be obtained successively. For $x=0$, (13.7) is $C(0; n; a_0, a_1, \cdots)=a_0C(0; n-1; a_0, a_1, \cdots)$. The solution of this difference equation in n, subject to the initial condition $C(0; 1; a_0, a_1, \cdots)=a_0$, is $C(0; n; a_1, a_2, \cdots)=a_0^n$. For $x=1$, (13.7) is $C(1; n; a_0, a_1, \cdots)-a_0C(1; n-1; a_0, a_1, \cdots)=a_1C(0; n-1; a_0, a_1, \cdots)$, a difference equation in n with the right-hand member a known function. The solution which satisfies the initial condition $C(1; 1; a_0, a_1, \cdots)=a_1$ is $C(1; n; a_0, a_1, \cdots)=na_0^{n-1}a_1$. If we assume that $C(0; n; a_0, a_1, \cdots), \cdots, C(x-1; n; a_0, a_1, \cdots)$ have been determined, then (13.7) gives the difference equation in n

$$C(x; n; a_0, a_1, \cdots) - a_0C(x; n-1; a_0, a_1, \cdots) = \sum_{i=1}^{x} a_i C(x-i; n-1; a_0, a_1, \cdots), \tag{13.9}$$

in which the right-hand member is a known function, for the determination of $C(x; n; a_0, a_1, \cdots)$. A complete induction then shows that all coefficients $C(x; n; a_0, a_1, \cdots)$ can be determined by solving successively the difference equations (13.9) subject to the initial condition $C(x; 1; a_0, a_1, \cdots)=a_x$.

14. Properties of the coefficients in the finite case. If $a_k\neq 0$, $a_i=0$ for $i>k$, then $C(x; n; a_0, \cdots, a_k)=0$ for $x>nk$. More precisely, it can be shown that $C(x; n; a_0, \cdots, a_k)$ has the factor n for $x>0$, the factor $n(n-1)$ for $x>k$, the factor $n(n-1)(n-2)$ for $x>2k$, and so on.

If the relation

$$C(x; n; a_0, \cdots, a_k) = C(nk-x; n; a_0, \cdots, a_k), \qquad x=0, 1, \cdots, nk, \tag{14.1}$$

holds for $n=1$, that is, if $a_x=a_{k-x}$ for $x=0, 1, \cdots, k$, then the same relation holds for every value of n. For, because of the assumed symmetry, $[t^k(a_0+a_1t^{-1}+a_2t^{-2}+\cdots+a_kt^{-k})]^n$ equals $t^{nk}\sum_0^{nk}C(x; n; a_0, \cdots, a_k)t^{-x}$ $=\sum_0^{nk}C(x; n; a_0, \cdots, a_k)t^{nk-x}$ on the one hand and $\sum_0^{nk}C(x; n; a_0, \cdots, a_k)t^x$

on the other. Equation (14.1) then follows by equating coefficients of t^x in the last two expressions written.

By multiplying the expansions of $(a_0+a_1t+\cdots+a_kt^k)^m$ and $(a_0+a_1t+\cdots+a_kt^k)^n$, it is found that

$$C(x; m+n; a_0, \cdots, a_k) = \sum_{i+j=x} C(i; m; a_0, \cdots, a_k)C(j; n; a_0, \cdots, a_k). \tag{14.2}$$

If $m=n$, $x=nk$, and the coefficients satisfy the symmetry relation (14.1), then

$$C(nk; 2n; a_0, \cdots, a_k) = \sum_{x=0}^{nk} [C(x; n; a_0, \cdots, a_k)]^2. \tag{14.3}$$

This equation in the case $k=1$ states a familiar property of the binomial coefficients.

15. Some special cases. The most interesting special case results from setting $a_i=1$, $i=0, 1, \cdots, k$, and $a_i=0$ for $i>k$. The values of $C(x; n; 1_0, \cdots, 1_k)$ for $k=1, 2, 3$ and $x=0, \cdots, 6$ are given in Table V, where the symbols denote binomial or multinomial coefficients as explained by (13.2).

TABLE V. Values of $C(x; n; 1_0, \cdots, 1_k)$ for $k=1, 2, 3$ and $x=0, 1, \cdots, 6$

x	$k=1$	$k=2$	$k=3$
0	$\binom{n}{0}$	$\binom{n}{0}$	$\binom{n}{0}$
1	$\binom{n}{1}$	$\binom{n}{1}$	$\binom{n}{1}$
2	$\binom{n}{2}$	$\binom{n}{2}+\binom{n}{1}$	$\binom{n}{2}+\binom{n}{1}$
3	$\binom{n}{3}$	$\binom{n}{3}+\binom{n}{1,1}$	$\binom{n}{3}+\binom{n}{1,1}+\binom{n}{1}$
4	$\binom{n}{4}$	$\binom{n}{4}+\binom{n}{2,1}+\binom{n}{2}$	$\binom{n}{4}+\binom{n}{2,1}+\binom{n}{2}+\binom{n}{1,1}$
5	$\binom{n}{5}$	$\binom{n}{5}+\binom{n}{3,1}+\binom{n}{1,2}$	$\binom{n}{5}+\binom{n}{3,1}+\binom{n}{1,1}+\binom{n}{2,1}+\binom{n}{1,1}$
6	$\binom{n}{6}$	$\binom{n}{6}+\binom{n}{4,1}+\binom{n}{2,2}+\binom{n}{3}$	$\binom{n}{6}+\binom{n}{4,1}+\binom{n}{2,2}+\binom{n}{3}+\binom{n}{1,1,1}+\binom{n}{3,1}+\binom{n}{2}$

For the case $k=1$ the triangle (13.8) is the familiar Pascal Triangle. The generalized Pascal Triangles shown in Table VI result in the cases $k=2$ and $k=3$.

TABLE VI. Generalized Pascal Triangles for $C(x; n; 1_0, \cdots, 1_k)$ in the Cases $k=2, 3$

n	$k=2$	$k=3$
0	1	1
1	1 1 1	1 1 1 1
2	1 2 3 2 1	1 2 3 4 3 2 1
3	1 3 6 7 6 3 1	1 3 6 10 12 12 10 6 3 1
4	1 4 10 16 19 16 10 4 1	1 4 10 20 31 40 44 40 31 20 10 4 1
·	· · ·

The recurrence relations (13.7) supply a simple means of writing out these triangles: for $k=2$ ($k=3$) any number in the triangle is equal to the sum of three (four) numbers in the row above it, namely, the one above it and the two (three) to its left.

By examining the solutions of the difference equations (13.9), or by applying properly the argument that led to (13.6), it can be shown that

$$C(x; n; 1_0, \cdots, 1_k) = \frac{n(n+1)\cdots(n+x-1)}{x!}, \qquad x = 0, 1, \cdots, k.$$

Equation (13.1) with $t=1$ gives $(k+1)^n = \sum_0^{nk} C(x; n; 1_0, \cdots, 1_k)$.

16. Applications to probability distributions. The solution of the problem stated in section 12 can now be described. If y is a random variable which takes on the values $0, 1, \cdots, k$ with probabilities $p_0, p_1, \cdots, p_k$, then $x = y + \cdots + y$ (n summands) is a random variable which takes on the values $0, 1, \cdots, nk$, and the probability that it takes on the value x is $C(x; n; p_0, \cdots, p_k)$. Let $\bar{x}_n$ and σ_n denote, for fixed k, the mean and standard deviation of the distribution $C(x; n)$, the latter being written for simplicity in place of $C(x; n; p_0, \cdots, p_k)$. Then*

$$\bar{x}_n = n\bar{x}_1 \tag{16.1}$$

$$\sigma_n^2 = n\sigma_1^2. \tag{16.2}$$

From the definition of $\bar{x}_n$ and (14.2) it follows that

$$\begin{aligned}\bar{x}_n = \sum_{x=0}^{nk} xC(x; n) &= \sum_{x=0}^{nk} \sum_{i+j=x} (i+j)p_i C(j; n-1) \\ &= \sum_{i=0}^{k} ip_i \sum_{j=0}^{(n-1)k} C(j; n-1) + \sum_{i=0}^{k} p_i \sum_{j=0}^{(n-1)k} jC(j; n-1) \\ &= \bar{x}_1 \cdot 1 + 1 \cdot \bar{x}_{n-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma_n^2 + \bar{x}_n^2 = \sum_{x=0}^{nk} x^2 C(x; n) &= \sum_{x=0}^{nk} \sum_{i+j=x} (i+j)^2 p_i C(j; n-1) \\ &= \sum i^2 p_i \sum C(j; n-1) + 2\sum ip_i \sum jC(j; n-1) \\ &\quad + \sum p_i \sum j^2 C(j; n-1) \\ &= (\sigma_1^2 + \bar{x}_1^2)\cdot 1 + 2\bar{x}_1 \cdot \bar{x}_{n-1} + 1\cdot(\sigma_{n-1}^2 + \bar{x}_{n-1}^2); \\ \sigma_n^2 &= \sigma_1^2 + \sigma_{n-1}^2.\end{aligned}$$

From these relations (16.1) and (16.2) follow by induction.

Consider an example. Suppose that n symmetrical dice are thrown, each die having $k+1$ faces numbered $0, 1, \cdots, k$. The problem is to find the probability

* The statement concerning the standard deviation was added at the suggestion of the referee.

distribution of the sum x of the values which appear on the n dice. In this case $p_i=1/(k+1)$. From (13.3) it follows that

$$C(x; n; p_0, \cdots, p_k) = \frac{C(x; n; 1_0, \cdots, 1_k)}{(k+1)^n}, \qquad p_i = \frac{1}{k+1},$$

for $x=0, 1, \cdots, nk$. The results in section 14 show that the probability distribution of x is symmetrical. According to the results in section 15, the distribution is similar in many respects to the binomial distribution. It follows both from the symmetry and from (16.1) that $\bar{x}=nk/2$; and from (16.2) that $\sigma^2=nk(k+2)/12$.

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THE FERMAT AND HESSIAN POINTS OF A TRIANGLE

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1. Introduction. The properties of the triangle which are presented here are proved analytically in the Morley *Inversive Geometry*.* The properties seem so noteworthy as to demand straightforward proofs by the methods of modern geometry, since, to the author's knowledge, the relations are neither mentioned nor proved in any of the standard works on the subject. The paper will confine itself to the discussion of those properties which are less familiar, it being assumed that the reader is acquainted with certain terms and theorems which are standard topics in modern geometry.

2. The Fermat points. The first points to be considered are those known as the Fermat points.† These may be defined in the following manner:

Consider a triangle $A_1A_2A_3$ with equilateral triangles $A_2A_3P_1$, $A_3A_1P_2$, $A_1A_2P_3$ described externally on its sides. Then it is known that the lines A_1P_1, A_2P_2, A_3P_3 are concurrent at a point F, which is known as a Fermat point of the triangle. Similarly, if equilateral triangles $A_2A_3P_1'$, $A_3A_1P_2'$, $A_1A_2P_3'$ are constructed internally on the sides of $A_1A_2A_3$ ($A_1A_2A_3$ being, in this case, non-equilateral) the lines A_1P_1', A_2P_2', A_3P_3' are concurrent at a second Fermat point, F'.

* Morley, Frank, and Morley, F. V.: Inversive Geometry, Ginn and Co., 1933.

† Also known as *isogonic centers*. Johnson, R. A.: Modern Geometry. Houghton Mifflin Co., 1929.

The Fermat points are characterized by the fact that the segments A_iP_i are equal and intersect at angles of 60°, and the segments A_iP_i' are equal and intersect at angles of 60°. The points are named from the fact that F is (for triangles having no angle exceeding 120°) the solution to Fermat's problem: to find a point the sum of whose distances from the vertices of a triangle is a minimum.

Let M_1 be the midpoint of A_2A_3 and let N_1 and N_1' be the centers of the equilateral triangles $A_2A_3P_1$ and $A_2A_3P_1'$ respectively. The centroid, G, of $A_1A_2A_3$ lies on A_1M_1, and $A_1G=2GM_1$. Then, since M_1 is the midpoint of P_1P_1', it is clear that G is also the centroid of $A_1P_1P_1'$, and therefore divides the median of this triangle drawn from P_1 in the ratio 2:1. Thus since $P_1N_1'=2N_1'P_1'$, GN_1' is parallel to A_1P_1'. Similarly it may be shown that GN_1 is parallel to A_1P_1.

Let N_1G and $N_1'G$ be produced to intersect A_1P_1' and A_1P_1 at R_1 and R_1' respectively. Then, since G is the centroid of $A_1P_1P_1'$, N_1R_1 and $N_1'R_1'$ are bisected at G, and circles may be described with centers at G and diameters N_1R_1 and $N_1'R_1'$. Let these circles respectively intersect A_1P_1' and A_1P_1 again in points E' and E. Now, since $N_1'E$ is perpendicular to A_1P_1 and N_1E' is perpendicular to A_1P_1', it follows that the circles circumscribing triangles $A_2A_3P_1$ and $A_2A_3P_1'$ pass, respectively, through E and E'. From previous considerations, therefore, we see that E and E' coincide with the Fermat points F and F' of $A_1A_2A_3$. In addition, since GN_1 and GN_1' are parallel to A_1F and A_1F', it follows that

$$\sphericalangle FA_1F' = \sphericalangle GFA_1 = \sphericalangle A_1F'G = \tfrac{1}{3}\sphericalangle FGF', \pmod{\pi}.$$

Summarizing we have: *GN_i and GN_i' are parallel to A_iP_i and A_iP_i' respectively; the points N_i and N_i' are the vertices of two equilateral triangles (with sides perpendicular to A_iF and A_iF') inscribed in the circles with centers at G and radii equal to GF' and GF respectively; and*

$$\sphericalangle FA_iF' = \sphericalangle GFA_i = \sphericalangle A_iF'G = \tfrac{1}{3}\sphericalangle FGF', \pmod{\pi}.$$

3. The Hessian points. Closely associated with the Fermat points are their isogonal conjugates, known as the Hessian points.* Let these points be designated by H and H', H being the isogonal conjugate of F. The sides of the pedal triangle $H_1H_2H_3$ of H are perpendicular to A_1F, A_2F, A_3F; the pedal triangle is therefore equilateral, its center being the midpoint, Q, of HF. Similarly, the pedal triangle $H_1'H_2'H_3'$ of H' is equilateral, its sides being perpendicular to A_1F', A_2F', A_3F', and its center at Q', the midpoint of $H'F'$.

Since the sides of the equilateral triangles $N_1N_2N_3$ and $H_1H_2H_3$ are respectively perpendicular to A_1F, A_2F, A_3F, the two triangles are homothetic. Let K be the center of similitude. Since, in the similitude, the center G of $N_1N_2N_3$ corresponds to the center Q of $H_1H_2H_3$ and the point H (the point of concurrence of perpendiculars at H_1, H_2, H_3 to the sides of $A_1A_2A_3$) corresponds to the circumcenter, O, of $A_1A_2A_3$, it follows that K must lie on OH and GQ, and that HF is parallel to the Euler line OG. Furthermore, since the distances of N_1, N_2, N_3 from the corresponding sides of $A_1A_2A_3$ are proportional to these sides, the

* Also known as *isodynamic points*. Johnson, *ibid.*

distances from K to the sides of $A_1A_2A_3$ are also proportional to these sides. K is therefore the symmedian point of $A_1A_2A_3$.* Similar considerations show that $H_1'H_2'H_3'$ is homothetic to $N_1'N_2'N_3'$, the center of similitude again being the symmedian point K. Thus K lies on OH' and on GQ', and $H'F'$ is parallel to OG. Now it is known that the points O, H, K, H' of any triangle form a harmonic set, so that G, Q, K, Q' must also be harmonic, from which it is easy to see that G, F, H' are collinear, as also are G, F', H and F, F', K. Summarizing the results we have:

In any triangle the line joining a Fermat point with the corresponding Hessian point is parallel to the Euler line; the line joining a Fermat point with the non-corresponding Hessian point passes through the centroid; the line joining the Fermat points intersects the line joining the Hessian points at the symmedian point.

4. Distance relations. Let a designate the distance GF' and let b designate the distance GF. Then

$$A_1P_1 = A_2P_2 = A_3P_3 = 3GN_1 = 3a,$$
$$A_1P_1' = A_2P_2' = A_3P_3' = 3GN_1' = 3b.$$

Applying the law of cosines to triangle $A_1A_2P_1$ we find

$$\begin{aligned} 9a^2 &= a_1^2 + a_3^2 - 2a_1a_3\cos(\alpha_2 + \pi/3) \\ &= a_1^2 + a_3^2 - a_1a_3\cos\alpha_2 + a_1a_3\sqrt{3}\sin\alpha_2 \\ &= \tfrac{1}{2}(2a_1^2 + 2a_3^2 - 2a_1a_3\cos\alpha_2) + a_1a_3\sqrt{3}\sin\alpha_2 \\ &= \tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) + 2\sqrt{3}\,\Delta, \end{aligned}$$

where Δ is the area of $A_1A_2A_3$. Similarly

$$9b^2 = \tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) - 2\sqrt{3}\,\Delta.$$

Now, let K_1 be the foot of the perpendicular from K to A_2A_3. Then†

$$KK_1 = 2a_1\Delta/(a_1^2 + a_2^2 + a_3^2).$$

Also

$$M_1N_1 = a_1\sqrt{3}/6.$$

Thus, if R is the ratio of similitude of $H_1H_2H_3$ to $N_1N_2N_3$,

$$\begin{aligned} R &= KK_1/(M_1N_1 + KK_1) \\ &= [2\Delta/(a_1^2 + a_2^2 + a_3^2)]/[\sqrt{3}/6 + 2\Delta/(a_1^2 + a_2^2 + a_3^2)] \\ &= 2\sqrt{3}\,\Delta/[\tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) + 2\sqrt{3}\,\Delta] \\ &= 2\sqrt{3}\,\Delta/9a^2. \end{aligned}$$

* Johnson, *ibid.*, Arts. 342, 490. K must be either the symmedian point or an exsymmedian point; but no one of the exsymmedian points lies on OHH'.

† Johnson, *ibid.*, Art. 342.

Similarly, if R' is the ratio of similitude of $H_1'H_2'H_3'$ to $N_1'N_2'N_3'$,

$$R' = -2\sqrt{3}\,\Delta/9b^2.$$

Then, from similar triangles,

$$GH/GF' = HQ/H'Q' = HK/H'K = (HK/OK)/(H'K/OK) = R/R' = b^2/a^2,$$

whence

$$GH = b^2/a.$$

Similarly

$$GH' = a^2/b.$$

Therefore: *H is the inverse of F' with respect to the circle circumscribing $N_1'N_2'N_3'$ and H' is the inverse of F with respect to the circle circumscribing $N_1N_2N_3$.*

Since H and F are isogonal conjugates

$$\sphericalangle A_3A_1H = \sphericalangle P_1A_1A_2.$$

and

$$\sphericalangle A_1A_3H = \sphericalangle FA_3A_2 = \sphericalangle A_1P_1A_2,$$

so that triangles A_1HA_3 and $A_1A_2P_1$ are similar, and $A_1H/a_2 = a_3/3a$, whence

$$A_1H = a_2a_3/3a.$$

For like reasons

$$A_1H' = a_2a_3/3b,$$

and it follows that

$$A_1H/A_1H' = b/a.$$

Since the same reasoning holds for the other vertices we have: *The circumcircle is the locus of a point whose distances from the Hessian points are in the constant ratio b/a.*

And since

$$GH/GH' = b^3/a^3$$

we have: *The ratio of the distances of the Hessian points to the centroid equals the cube of the ratio of the distances from the Hessian points to any vertex.*

Also, $\sphericalangle HA_iH' = \sphericalangle FA_iF'$, whence: *The angle subtended at the Hessian points by any vertex equals one third the angle (mod π) subtended at the Hessian points by the centroid.*

These relations are noteworthy since they lead to a geometric method for the solution of a cubic equation.*

* Morley and Morley, *ibid.*, Art. 42.

The distance between the Hessian points may be expressed in terms of the sides of the triangle by considering the triangles $N_1N_1'G$ and $H'HA_1$. These triangles are similar since $\sphericalangle N_1'GN_1 = \sphericalangle HA_1H'$, and $A_1H/A_1H' = b/a = GN_1'/GN_1$. Therefore

$$HH'/N_1N_1' = A_1H/b \quad \text{and} \quad HH'/N_1N_1' = A_1H'/a,$$

whence

$$(HH')^2/(N_1N_1')^2 = (A_1H)(A_1H')/ab = a_2^2a_3^2/9a^2b^2,$$

or

$$HH'/N_1N_1' = a_2a_3/3ab.$$

Hence, since $N_1N_1' = a_1\sqrt{3}/3$,

$$HH' = a_1a_2a_3\sqrt{3}/9ab.$$

Substituting

$$3a = [\tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) + 2\sqrt{3}\,\Delta]^{1/2},$$
$$3b = [\tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) - 2\sqrt{3}\,\Delta]^{1/2},$$

we finally have

$$HH' = 2a_1a_2a_3\sqrt{3}/[a_1^2 + a_2^2 + a_3^2 - 48\Delta^2]^{1/2}.$$

If $\sphericalangle HGH' = 3\phi$, then $\sphericalangle N_i'GN_i = \phi$, (mod $\pi/3$). We may always so letter the given triangle such that

$$\sphericalangle N_1'GN_1 = \phi, \qquad \sphericalangle N_2'GN_2 = \pi/3 - \phi, \qquad \sphericalangle N_3'GN_3 = 2\pi/3 - \phi.$$

Then, by applying the law of cosines to the triangles $N_i'GN_i$, we find

$$\cos^2\phi = (a_2^2 + a_3^2 - 2a_1^2)^2/(a_1^2 + a_2^2 + a_3^2 - 48\Delta^2),$$
$$\cos^2(\pi/3 - \phi) = (a_3^2 + a_1^2 - 2a_2^2)^2/(a_1^2 + a_2^2 + a_3^2 - 48\Delta^2),$$
$$\cos^2(2\pi/3 - \phi) = (a_1^2 + a_2^2 - 2a_3^2)^2/(a_1^2 + a_2^2 + a_3^2 - 48\Delta^2).$$

In a similar manner, distances between other points in the figure may be expressed in terms of a, b, and the angle ϕ, and consequently in terms of the sides of the triangle itself, if desired.

A little joke (?) by the late David Eugene Smith. Ever so often a kind fate intervenes to make some error, to which all humans are prone, result in something amusing. In David Eugene Smith's monumental *Rara Arithmetica*, under the date 1537, Professor Smith listed as one of the arithmetical publications of that year the following: "L. Culman, 'Wie iung und alte Leut petten sollen,' Nurnberg, 8°; ib., 1537." The British Museum Catalogue of Printed Books gives this title, with one letter changed, "betten" (to pray), and the operation of "praying" is far from the American meaning of "petting" or "petten," and neither one is strictly arithmetical.—*L. C. Karpinski.*

A VECTOR INTERPRETATION OF THE DERIVATIVE CIRCLE

H. R. PYLE AND BEVERLY M. BARKER, Whittier College

1. Real functions. Let $w(x, y)$ represent a real function of x and y, continuous and with continuous first partial derivatives in a given region. Then $w=c$ is a contour line which we shall call the curve C (Fig. 1). Let R be a point on C whose coordinates are x and y. Then $z=x+iy$ may be considered the position vector of R.

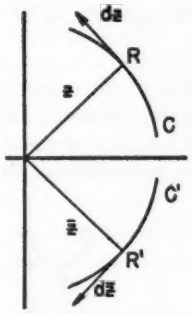

Fig. 1

Let $w'=w(x, -y)$ represent a second real function related to w, which is also continuous and whose partial derivatives are continuous in the region under consideration. Then $w'=c'$ is a contour line for w' which we shall call the curve C'. Let R', whose coordinates are x and $-y$, be the point on C' corresponding to R on C. Then $\bar{z}=x-iy$ is the position vector of R'. C' is symmetric to C with respect to the x-axis. If C is symmetric with respect to the x-axis, C' coincides with C.

The derivative circle for dw/dz may be found by the well-known process worked out by Kasner [1]. As noted by Hedrick [4],

$$\frac{dw}{dz} = D + Pe^{-2i\theta}$$

where

$$D = \frac{\partial w}{\partial z} = \tfrac{1}{2}(w_x - iw_y), \qquad P = \frac{\partial w}{\partial \bar{z}} = \tfrac{1}{2}(w_x + iw_y).$$

Here D represents the vector from the origin to the center of the derivative circle and $Pe^{-2i\theta}$ represents the vector from the center to any point on the circle. Hence the magnitude of P is the radius of the circle. The coordinates of the center are $(w_x/2, -w_y/2)$ and the radius is

$$|P| = \tfrac{1}{2}\sqrt{w_x^2 + w_y^2}\,.$$

In the terminology of vector analysis the gradient of w at the point R is written $\nabla w = lw_x + mw_y$ where l and m are unit vectors along the x and y axes respectively. In the notation of complex variables $\nabla w = w_x + iw_y$. Thus $P = \frac{1}{2}\nabla w$.

THEOREM 1. *When w is a real function of x and y, $P = (\partial w/\partial \bar{z}) = \frac{1}{2}\nabla w$ is a vector along the gradient of the curve $w(x, y) = C$ whose magnitude is half that of the gradient.*

Similarly, the gradient of w' at R' is $\nabla w' = lw_x' + mw_y'$. But at R', $w_x' = w_x$, $w_y' = -w_y$. Therefore $\nabla w' = lw_x - mw_y$, or $\nabla w' = w_x - iw_y$. Hence $D = \frac{1}{2}\nabla w'$, and is represented by a vector along the gradient of C' whose magnitude is half that of the gradient.

THEOREM 2. *The position vector of the center of the derivative circle of dw/dz has a magnitude equal to half the magnitude of the gradient of w, and is parallel to the gradient of $w' = w(x, -y)$; i.e., it is normal to the curve $w' = c'$ at R'. The diameter of the circle is equal to the magnitude of the gradient.*

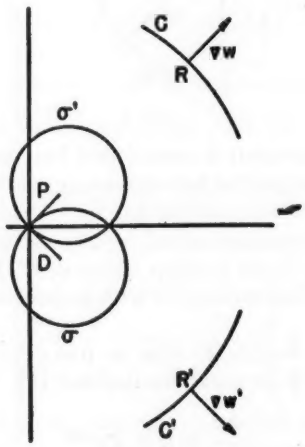

Fig. 2

In the same manner we find that

$$\frac{dw}{d\bar{z}} = P + De^{2i\theta}.$$

Therefore P is the position vector of the center of the circle and the magnitude of D is the radius. The coordinates of the center are $(w_x/2, w_y/2)$ and the radius is

$$|D| = \tfrac{1}{2}\sqrt{w_x^2 + w_y^2}.$$

Hence the position vector of the center has a magnitude equal to half the magnitude of the gradient and is parallel to the gradient of w; *i.e.*, it is normal to C at R (Fig. 2). The diameter is the same as the diameter of the circle representing dw/dz. The two circles are symmetrically placed relative to the x-axis and they both go through the origin. We shall call them conjugate derivative circles. In the figure, σ is the circle representing dw/dz; σ' represents $dw/d\bar{z}$.

$$D = \tfrac{1}{2}\nabla w', \qquad P = \tfrac{1}{2}\nabla w.$$

2. Complex functions. Let $F=u+iv$ be a function whose components $u(x, y)$ and $v(x, y)$ are real functions, continuous and with continuous first partial derivatives in the region under consideration. In 1927, Kasner [1] introduced the term *polygenic function* to denote a general complex function whose components do not satisfy, necessarily, the Cauchy-Riemann equations. Then $u=c_1$, $v=c_2$ are contour lines for u and v respectively. Call $u=c_1$ the curve C_1 and $v=c_2$ the curve C_2. Let the point R, with coordinates x and y, be their point of intersection. Then $z=x+iy$ is the position vector of R.

Consider the function $F'=u'+iv'$, where $u'=u(x, -y)$, $v'=v(x, -y)$, and call $u'=c_1'$ the curve C_1' and $v'=c_2'$ the curve C_2'. R', with coordinates x and $-y$, is the point of intersection of C_1' and C_2'. $\bar{z}=x-iy$ is its position vector.

Since

$$\begin{aligned} D(u) &= \tfrac{1}{2}(u_x - iu_y), \qquad D(v) = \tfrac{1}{2}(v_x - iv_y) \\ P(u) &= \tfrac{1}{2}(u_x + iu_y), \qquad P(v) = \tfrac{1}{2}(v_x + iv_y), \\ D(u) + iD(v) &= \tfrac{1}{2}[(u_x + v_y) + i(-u_y + v_x)] \\ P(u) + iP(v) &= \tfrac{1}{2}[(u_x - v_y) + i(u_y + v_x)]. \end{aligned}$$

Referring to Hedrick's paper [4], we see that

$$D(F) = \frac{\partial F}{\partial z} = D(u) + iD(v), \qquad P(F) = \frac{\partial F}{\partial \bar{z}} = P(u) + iP(v).$$

Kasner [2] calls $D(F)$ the *mean derivative* and $P(F)$ the *phase derivative.*

We have seen that $D(u)$ is a vector along the gradient of C_1' whose length is half the magnitude of the gradient of u and $D(v)$ is a vector along the gradient of C_2' whose length is half the magnitude of the gradient of v. If we interpret multiplication by i as counter-clockwise rotation through a right angle, $iD(v)$ is a vector tangent to C_2'.

THEOREM 3. *The mean derivative $D(F)=D(u)+iD(v)$, where $D(u)$ is one-half the gradient of C_1' (defined by $u(x, -y)=c_1'$), and $iD(v)$ is a vector tangent to C_2' (defined by $v(x, -y)=c_2'$), whose magnitude is equal to one-half the gradient of C_2'.*

The same interpretation holds for $P(F)$, referred to the contour lines C_1 and C_2.

$D(F)$ represents the position vector of the center of the derivative circle for dF/dz, and the magnitude of $P(F)$ is the radius (Fig. 3). Likewise $P(F)$ repre-

sents the position vector of the center of the circle for $dF/d\bar{z}$ and the magnitude of $D(F)$ is its radius. For the real function w, $|D(w)| = |P(w)|$. This makes the radii of the two circles σ and σ' equal, and causes both to pass through the origin. The equality does not hold for $|D(F)|$ and $|P(F)|$.

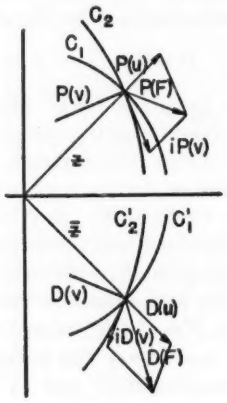

Fig. 3

When F is an analytic function of z, it is well-known that the contour lines C_1 and C_2 are orthogonal and the magnitudes of the gradients of u and v are equal. Furthermore it can be proved that the angle from the positive gradient of C_1 to the positive gradient of C_2 is counter-clockwise.

The angle from the positive gradient of $u'=u(x, -y)$ to the positive gradient of $v'=v(x, -y)$ is clockwise and equal to a right angle when $F=u+iv$ is an analytic function of z. The magnitudes of the gradients of u' and v' are equal.

Since $P(F)=P(u)+iP(v)$ and $P(u)$ and $iP(v)$ are collinear, of equal length, and oppositely directed when F is an analytic function of z, $P(F)=0$. Furthermore, since $D(F)=D(u)+iD(v)$, and $D(u)$ and $iD(v)$ are collinear, of equal length, and similarly directed, $D(F)=2D(u)=\nabla u'$.

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"I had forgotten that on the moon, with only an eighth part of the earth's mass and a quarter of its diameter, my weight was barely a sixth of what it was on earth. But now that fact insisted on being remembered." H. G. Wells, *The First Men in the Moon* (Error noted by MacKay in *Odd Numbers*).—*E. D. Schell.*

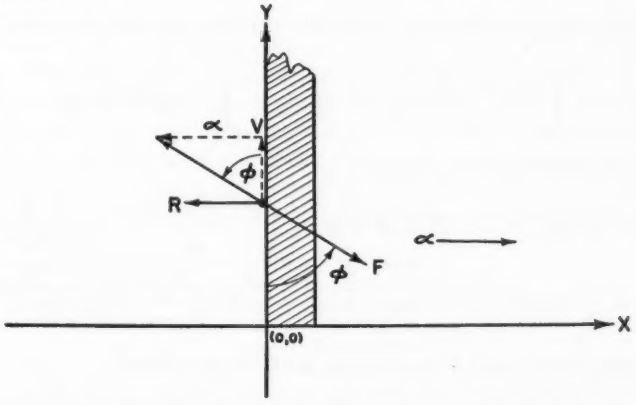
A PROBLEM IN FRICTION

F. A. VALENTINE, University of California at Los Angeles

The following elementary problem in mechanics has some pedagogical interest as well as physical interest. The author has asked various people to hazard a guess as to what the solution might be. In most cases the controversy aroused seemed to me to be of as much interest as the problem itself. For that reason the problem is stated as it originated in one of my classes in mechanics. The mathematical treatment is given first. A brief physical interpretation of the results is then given. The application mentioned at the end of the article was called to my attention by Prof. W. Mason.

Problem: A *rough* horizontal floor moves in a direction perpendicular to a *smooth* vertical wall, disappearing under the base of the wall. The velocity of the floor is a constant, α. A particle is projected with an initial velocity v_0 along the line of intersection of the floor and the wall. Determine the distance traveled by the particle, and the time required to travel this distance. (Assume both wall and floor to be of infinite extent.)

Solution: Choose a rectangular coordinate system (x, y, z) so that the line of intersection of the floor and wall coincides with the y-axis, and so that the vertical wall coincides with the (y, z) plane. Suppose the motion of the floor is in the direction of the positive x-axis. If $[x(t), y(t), 0]$ are the coordinates of the particle p, the initial conditions may be expressed in the form $x(0)=0$, $y(0)=0$, $x'(0)=0$, $y'(0)=v_0>0$.



The force of friction F of the floor acting on the particle p is opposite to the direction of the velocity of p *relative* to the floor. Let ϕ be the acute angle which this direction makes with the y-axis. The reaction of the wall is denoted by R. The following diagram illustrates the motion in the (x, y)-plane.

Furthermore $|F| = \mu Mg$, where μ = coefficient of friction, and M = mass of p. The equations of motion of p are

$$M\frac{d^2x}{dt^2} = -R + \mu Mg\sin\phi \equiv 0,$$

(1)

$$M\frac{d^2y}{dt^2} = -\mu Mg\cos\phi.$$

Clearly $x''(t)\equiv 0$, for if there exists an interval $t_1 \leqq t \leqq t_2$ such that $x(t_1)=0$, $x'(t_1)=0$, $x''(t)<0$, $(t_1<t\leqq t_2)$, we would have $x(t)<0$ and $x'(t)<0$ for $(t_1<t\leqq t_2)$. However, in this case $R\equiv 0$, $(t_1<t\leqq t_2)$, which would imply the contradictory result $x''(t)>0$, $(t_1<t\leqq t_2)$. Since $x(t)$ cannot be positive because of the wall, it follows that $x''(t)\equiv 0$.

Since

$$\cos\phi = \frac{y'}{\sqrt{y'^2+\alpha^2}}$$

the second of equations (1) has the form

$$\frac{d^2y}{dt^2} = \frac{-\mu g y'}{\sqrt{y'^2+\alpha^2}} \tag{2}$$

where $y'=dy/dt$. This is a special case of the differential equation $y''=f(y')$. For our purpose it is advisable to use a parametric solution of the form

$$t - t_0 = \int_{v_0}^{v}\frac{dp}{f(p)}, \qquad y - y_0 = \int_{v_0}^{v}\frac{p\,dp}{f(p)}, \qquad f(p)\not\equiv 0,$$

where $y'=v$, and $y''=(dv/dy)v$. Equation (2) then has the parametric solution

$$\mu g t = -\int_{v_0}^{v}\frac{\sqrt{p^2+\alpha^2}}{p}\,dp, \qquad \mu g y = -\int_{v_0}^{v}\sqrt{p^2+\alpha^2}\,dp, \qquad v\not\equiv 0.$$

These are expressible in the form

$$\mu g t = \sqrt{v_0^2+\alpha^2} - \sqrt{v^2+\alpha^2} + \alpha\log\left(\frac{\sqrt{v_0^2+\alpha^2}-\alpha}{v_0}\cdot\frac{v}{\sqrt{v^2+\alpha^2}-\alpha}\right),$$

(3)

$$\mu g y = \frac{1}{2}\left(v_0\sqrt{v_0^2+\alpha^2} - v\sqrt{v^2+\alpha^2}\right) + \frac{\alpha^2}{2}\log\left(\frac{v_0+\sqrt{v_0^2+\alpha^2}}{v+\sqrt{v^2+\alpha^2}}\right).$$

Observe that both t and y are defined for $0<v\leqq v_0$. Since

$$\lim_{v\to 0}\frac{v}{\sqrt{v^2+\alpha^2}-\alpha} = \infty,$$

$$\lim_{v\to 0}\frac{1}{v^2+\sqrt{v^2+\alpha^2}} = \frac{1}{\alpha}, \qquad \alpha\neq 0,$$

equations (3) imply that for $\alpha\neq 0$,

$$\lim_{\alpha \to 0} t = \infty,$$

$$\lim_{\alpha \to 0} y = \frac{1}{2\mu g}\left[v_0\sqrt{v_0^2+\alpha^2} + \alpha^2 \log\left(\frac{v_0+\sqrt{v_0^2+\alpha^2}}{\alpha}\right)\right]. \tag{4}$$

This *solution* of the problem is based on the assumption that the usual laws of friction hold.

It is interesting to observe that certain of these results can be obtained by an application of general existence theorems for differential equations.* Equation (2) can be written in the form

$$\frac{dv}{dt} = -\mu g \frac{v}{\sqrt{v^2+\alpha^2}} \equiv f(v), \qquad v(0) = v_0 > 0. \tag{5}$$

When $\alpha=0$, dv/dt is a negative constant in which case $v(t)\to 0$ for a finite value of t. When $\alpha\neq 0$, the boundedness of $|f(v)|$ for *all values* of v implies, by virtue of the existence theorems, that $v(t)$ is defined for all values of t. The solution $v(t)$ is decreasing. It cannot vanish for a *finite* value of t, otherwise the solution would not be *unique*, since $v(t)\equiv 0$ is also a solution. Moreover, if $v(t)\geqq c>0$ for all t, then (5) implies that $dv/dt \leqq -\mu g c(v_0^2+\alpha^2)^{-1/2}$. But this implies that $v(t)$ is tending to zero for a *finite* value of t, which is impossible by virtue of the preceding arguments. Hence $v(t)\to 0$ as $t\to\infty$, when $\alpha\neq 0$.

The physical significance of the above results appears plausible when one considers the fact, $\tan\phi = \alpha/v$. As $v\to 0$, the angle $\phi\to\pi/2$. Hence the vertical component of F, $F_1 \equiv |F|\cos\phi$, approaches zero as $v\to 0$, since $|F|$ is constant.

This type of problem arises in other places, and the principle is also put to practical use. In the checking of pressure gages, a dead weight tester† is often used. In this instrument a cylindrical piston which supports standardized weights is rotated about its axis of symmetry in order to reduce the *vertical* component of wall friction, thus allowing easier vertical movement. This is not solely a matter of static friction. Furthermore observe in the problem discussed here, that if a particle is at rest at the base of the wall, it can be set into motion more easily, if the floor is in motion. The larger the velocity α, the easier is the effort.

* This verification was suggested to me by the referee.

† Moyer, J. A., Power Plant Testing, 1913. McGraw-Hill, p. 17.

Tolstoy's modesty. "Had I been told when I was at the university that others understood the integral calculus while I did not, that would have touched my pride." Tolstoy.—*P. L. Chessin.*

Wolves. "Mathematicians are like lovers: grant a mathematician the least principle, and he will draw from it a consequence which you must also grant him, and from this consequence another." Fontenell.—*P. L. Chessin.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A REFERENCE

J. H. M. WEDDERBURN, Princeton University

Since writing my note *On Pascal's Theorem*, which appeared in the August-September issue of this MONTHLY, I have noticed that equation (1) of the lemma used is given by L. N. M. Carnot, *Géométrie de Position*, Paris, 1803, §236, p. 293.

I might have pointed out that the proof given applies to Pappus' Theorem and gives what seems to be the most elementary proof of it since it only involves Menelaus' Theorem. The case in which AP_1, BP_2, CP_3 meet in a point then leads to a number of interesting exercises in euclidean geometry. It is worth noticing also that the proof only requires four points on each line although the figure degenerates if there are fewer than five.

A PSYCHOLOGICAL GAME

N. S. MENDELSOHN, Queen's University

This contribution is the outcome of an informal discussion with Dr. I. Kaplansky last June at the Canadian Mathematical Congress.

The following game is considered. Players A and B each choose a positive integer simultaneously; the player whose number is the smaller scores one point, unless the other player chooses a number exactly one greater, in which case the latter scores two points. I have called this a psychological game because in practice each player attempts to guess his opponent's next move, and usually, after a short while, one of the players gains a psychological ascendancy and piles up a huge score, in spite of the fact that the game is equitable.

This game is subject to mathematical analysis. It can be shown that the best strategy for a player A is to choose numbers 1, 2, 3, 4, 5, with frequencies 1, 5, 4, 5, 1; the choice being made in random order. The strategy is best in the following sense; (*a*) there is no strategy which can beat it in the long run; (*b*) corresponding to any other strategy, a winning counter strategy can be devised. In particular, any strategy which incorporated the number six or higher numbers would be a losing strategy.

Firstly, a small generalization. Assume that the second player scores n points ($n>1$, but is not necessarily an integer) instead of two points in the case where he chooses an integer one greater than that of the first player. In connection with this game some of the problems of interest are: (1) for a given n, determine

if it exists, the number of integers necessary to incorporate into a best strategy (in the original game this is five); (2) determine the frequency with which the various integers should be played; (3) determine the largest number n for which a winning strategy can be devised using only the first $2r+1$ integers.

The problem is solved as follows. First, form A's matrix. This is the matrix (a_{uv}) where a_{uv} is the number of points A scores when A chooses u and B chooses v. (We keep score for A alone. If B wins we subtract the points from A's score.)

$$(a_{uv}) = \begin{vmatrix} 0 & -n & 1 & 1 & 1 & 1\cdots 1 & 1\cdots \\ n & 0 & -n & 1 & 1 & 1\cdots 1 & 1\cdots \\ -1 & n & 0 & -n & 1 & 1\cdots 1 & 1\cdots \\ -1 & -1 & n & 0 & -n & 1\cdots 1 & 1\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1\cdots & & & & \end{vmatrix}.$$

Suppose that a best strategy can be devised which incorporates only the first $2r+1$ integers. Let $p(i, r)$ be the relative frequency with which i should be played using this strategy. We are interested in the relative magnitude of the numbers $p(1, r), p(2, r), p(3, r), \cdots, p(2r+1, r)$. It can be seen from symmetry that $p(1, r)=p(2r+1, r)$, $p(2, r)=p(2r, r), \cdots, p(j, r)=p(2r+2-j, r)$. For convenience, we will replace $p(j, r)$ by $p(2r+2-j, r)$ whenever $j>r+1$. Since the game is equitable A's average score should be 0, whenever B plays any of $1, 2, \cdots, (2r+1)$; (if A could have an average score greater than 0, B by using the same strategy as A could also obtain an average score greater than 0, a contradiction). The condition that A's average score is 0, if B plays $1, 2, \cdots, (2r+1)$, leads to the following $(r+1)$ equations (actually $2r+1$ equations are obtained of which r are repeated once).

$$\begin{aligned} &-p(1, r) + (n-1)p(2, r) - 2p(3, r) - \cdots - 2p(r, r) - p(r+1, r) = 0 \\ &-(n+1)p(1, r) - p(2, r) + (n-1)p(3, r) - 2p(4, r) - \cdots - p(r+1, r) = 0 \\ &0p(1, r) - (n+1)p(2, r) - p(3, r) + (n-1)p(4, r) - \cdots - p(r+1, r) = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &0p(1, r) + 0p(2, r) + \cdots - (n+1)p(r-1, r) - p(r, r) + np(r+1, r) = 0. \end{aligned}$$

Furthermore, if a best strategy is actually attained, A's score should be positive if B plays $2r+2$ or higher. If B plays $2r+2$, one obtains as a condition for a positive score the inequality

$$(n-1)p(1, r) - 2p(2, r) - \cdots - 2p(r, r) - p(r+1, r) < 0.$$

If B plays $2r+3$ or more, A's average score is positive without condition.

The above condition gives the maximum n for which a best strategy using $1, 2, 3, \cdots, (2r+1)$ is possible. A lower bound for n is obtained by assuming

that no best strategy can be based on fewer than the first $2r+1$ integers. The number $p(i, r)$ is $(-1)^{i+1}$ times the determinant obtained by removing the ith column from the following r-rowed, $(r+1)$-columned matrix:

$$\begin{vmatrix} -1 & (n-1) & -2 & -2 & -2 & -2 \cdots -2 & -1 \\ -(n+1) & -1 & (n-1) & -2 & -2 & -2 \cdots -2 & -1 \\ 0 & -(n+1) & -1 & (n-1) & -2 & -2 \cdots -2 & -1 \\ 0 & 0 & -(n+1) & -1 & (n-1) & -2 \cdots -2 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots & & & -(n+1) \; -1 & n \end{vmatrix}.$$

By expanding $p(i, r)$ along the top row, difference equations for $p(i, r)$ are obtained, *e.g.*:

$$p(1, r) = (n-1)p(1, r-1) - 2p(2, r-1) - \cdots - 2p(r-1, r-1) - p(r, r-1);$$

$$p(2, r) = p(1, r-1) + 2(n+1)p(1, r-2) + \cdots + 2(n+1)p(r-2, r-2) + (n+1)p(r-1, r-2);$$

$$p(3, r) = p(2, r-1) + (n^2-1)p(1, r-2) - 2(n+1)^2p(1, r-3) - \cdots - 2(n+1)^2p(r-3, r-3) - (n+1)^2p(r-2, r-3); \textit{ etc.}$$

The first of these relations yields a very elegant result. On examining the condition for a best strategy based on the first $(2r+1)$ integers, it is seen that this reduces to $p(1, r+1)<0$. This sets a least upper bound for the value of n. The following short table gives results for the first few cases:

r	*Condition on* n	$p(i, r)$
1	$n^2-n-1<0$	$p(1, 1)=n$, $p(2, 1)=1$
2	$n^3-3n^2-4n-1<0$	$p(1, 2)=n^2-n-1$, $p(2, 2)=2n+1$, $p(3, 2)=n^2$
3	$n(n^3-6n^2-9n-3)<0$	$p(1, 3)=n^3-3n^2-4n-1$, $p(2, 3)=n(3n+2)$, $p(3, 3)=n(n^2-n-1)$, $p(4, 3)=4n^2+4n+1$.

The game can be generalized in other ways (*e.g.*, by a more generalized scoring system, the score alloted being a function of the difference between the numbers chosen by the players). The mathematical treatment of these generalizations would follow the same lines. There is one point, however, worth noting. The author has avoided the case of strategies based on the use of an even number of integers. This case is fundamentally more difficult. The difficulty arises from the fact that a skew-symmetric matrix is always of even rank. The author throws open this problem to any who may be interested.

SOME SPHERES ASSOCIATED WITH A TETRAHEDRON*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. A known property. In a triangle ABC the lines joining the vertices A, B, C to the points of contact D, E, F of the inscribed circle with the sides BC, CA, AB respectively, are concurrent at the *Gergonne point* of the triangle, and the circles inscribed in the triangles BAD and CAD touch each other on the line AD, the circles inscribed in the triangles CBE and ABE touch each other on the line BE, and the circles inscribed in the triangles ACF and BCF touch each other on the line CF.

2. A space analogy. In any tetrahedron $T \equiv ABCD$, the point P whose barycentric coordinates are proportional to the trigonometric tangents of the half-angles at the vertices A, B, C, D of the cones of revolution inscribed in the trihedral angles (A), (B), (C), (D) is comparable to the Gergonne point of a triangle.†

Let A_1, B_1, C_1, D_1 be the points of contact of the inscribed sphere (I) of T on the faces BCD, CDA, DAB, ABC. If we mark points β, γ, δ on A_1B, A_1C, A_1D such that $A_1\beta = A_1\gamma = A_1\delta = s$, say, and let the perpendiculars at β, γ, δ to the plane BCD cut IB, IC, ID in ω_b, ω_c, ω_d, then *the spheres with centers* $\omega_b, \omega_c, \omega_d$ *and radii* $\omega_b\beta, \omega_c\gamma, \omega_d\delta$ *have for radical axis the fixed line* A_1P, *no matter what be the length* s.‡ Therefore, if one joins the Gergonne point P of T to the contacts A_1, B_1, C_1, D_1 of the inscribed sphere, the planes bisecting the angles PA_1B, PA_1C, PA_1D respectively cut IB, IC, ID at the centers ω_b', ω_c', ω_d' of three spheres inscribed in the trihedral angles (B), (C), (D) and touching the line PA_1 in the same point Q, for, by an easy indirect argument, PA_1 is the radical axis of the spheres (ω_b'), (ω_c'), (ω_d'). The same property applies to the triples of spheres inscribed in the trihedral angles $[(C), (D), (A)]$, $[(D), (A), (B)]$, $[(A), (B), (C)]$ which touch the lines PB_1, PC_1, PD_1 respectively at common points. Thus one finds a solution to the problem: *In three of the trihedral angles of a tetrahedron* $ABCD$, *to inscribe spheres tangent to a common line at a common point.*

In an isogonic tetrahedron T, the lines PA_1, PB_1, PC_1, PD_1 pass through the vertices A, B, C, D and the triples of spheres (ω_b'), (ω_c'), (ω_d'), $\cdots$, touch the lines AA_1, BB_1, CC_1, DD_1 respectively at common points.

Note. The normal coordinates of the point P defined by means of the barycentric coordinates of R. Bouvaist (*loc. cit.*) are proportional to the radii of the circles circumscribing the faces of T, and the point P is also called the *second Lemoine point of* T.§

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* Translated from the French by Howard Eves.

† R. Bouvaist, supplement by V. Thébault, Mathesis, 1941, p. 19.

‡ R. Bouvaist and V. Thébault, Comptes-Rendus des séances de l'Académie des sciences de Paris, 1940, p. 378.

§ V. Thébault, Annals de la Soc. Scient. de Bruxelles, 1922, pp. 173–177.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1944–45

Mathematics Society, Brooklyn College

The purpose of the Society is to create and maintain an interest in mathematics. Special activities, such as the publication of the *Math Mirror* (a student magazine on mathematics with an issue of 1,000 copies) and the Integration Contests, assist us in furthering this aim. But it is at the regular meetings, where the faculty and student members lecture, that we attain our goal. Lectures during the Fall semester 1944 were as follows:

Euler squares and Latin squares, by Professor Harris MacNeish
Methods of summation of finite series, by Jack Melnick
Systems of linear equations, by Julius Jackson
Approximations to the $\sqrt{2}$, by Professor James Singer
An algebra of plane geometry, by Prosessor Walter Prenowitz
Probability, by Bernard Seckler
Self-inductive theorems, by Professor Roger Johnson
The E function, by Ezra Krendel.

During the Spring semester, lectures were delivered as follows:

The hyperboloid of one sheet, by Professor Harris MacNeish
The Gamma function and Dirichlet integrals, by Jack Melnick
History and concepts of the calculus, by Professor Carl Boyer
Crytography and cryptanalysis, by Professor Jack M. Wolfe
Applications of mathematics to chemistry, by Professor Martin Meyer.

The twenty-ninth semi-annual Integration Contest, held in the Fall of 1944, resulted in victory for Professor Johnson's team. Individual honors went to Violet Bushwick. The thirtieth Contest, held in the Spring of 1945, resulted in victory for Professor Merle Bishop's first team. Individual honors went to Abigail Wishneff. During the Spring semester, the Mathematics Society published its thirteenth annual issue of the *Math Mirror*, edited by Jack Melnick, Editor-in-chief, and Bernard Seckler and Jerome M. Rehr, Associate Editors.

The officers of the Fall semester were: President, Julius Jackson; Vice-President, Ezra Krendel; Secretary, Marion Eisen; Treasurer, Jerome Rehr. The officers for the Spring semester were: President, Jack Melnick; Vice-President, Bernard Seckler; Secretary, Marion Eisen; Treasurer, Jerome Rehr. The Faculty Adviser for the Mathematics Society was Professor James Singer. The Faculty Adviser for the *Math Mirror* was Professor Harris MacNeish.

Kappa Mu Epsilon, Kansas State Teachers College, Pittsburg

Kansas Alpha Chapter held open meetings in the homes of the departmental staff members. At each meeting two or three students and one faculty member gave papers or talks on mathematical topics. Music numbers or readings were given by members or invited guests. Refreshments were served by the hostess. Student topics were:

Mathematical symbols, by Jack Marshall
Mathematics in the schools of England, by Dorothy Bradshaw
Proofs of the Pythagorean theorem, by Ralph Newman
Euclid, by Christine Carpenter
Mathematical recreations, by Miss Billie Shultz
Mathematics in the air age, by Bill Seal
Mathematics of the early American Indians, by Ann Benny
Mathematical societies and publications, by Bob Besser
David Eugene Smith, by Hattie Highfill
Surveying in ancient times, by Jack Hedgecock.

Papers presented by the faculty were:

Mathematical contests, by Dr. R. G. Smith
Quadrature of plane areas, by Professor J. A. G. Shirk
Addition and subtraction logarithms, by Professor F. C. German
United States land surveys, by Professor L. E. Curfman
Electronics and radar, by Professor W. H. Matthews of the Physics Department.

Three initiation services were held during the year for seventy-five students. Mrs. Dorothy Bradshaw and Dorothy Campion received *Kappa Mu Epsilon* pins for having the highest scholarship records of seniors majoring in mathematics. Chapter officers for 1944–45 were: President, Mrs. Dorothy Bradshaw; Vice-President, Dorothy Bernhardt; Secretary, Nancy Horton; Treasurer, Ann Benny; Corresponding Secretary, Professor W. H. Hill; Sponsor, Professor J. A. G. Shirk.

Pi Mu Epilson, University of Kentucky

During the year 1944–45 the activities of the Club were as follows:

Vectors and complex numbers, by Professor C. G. Latimer
Parametric representation, by Professor H. H. Downing
Gauss' theorem and an application, by Miss Mary Ann Macke, graduate student.

The annual picnic and initiation was held at Castlewood Barn. Eighteen members and friends were in attendance, and Mr. T. M. Hahn, a physics major, was initiated.

During the year the club presented to the mathematics library over fifty dollars worth of books.

Officers for 1945–46 are as follows: Director, Professor Flora LeStourgeon; Vice-Director, Professor H. H. Downing; Secretary-treasurer, Mrs. Lydia Fischer; Librarian, Professor M. C. Brown.

Mathematics Club, University of Buffalo

The Mathematics Club of the University of Buffalo tries to make its meetings interesting not only to those people who are in the field or who are about to enter it, but also to those who are comparatively unfamiliar with it. During the year 1944–45, as always, we have attempted to fulfill our purpose by presenting diversified programs showing new views of mathematics, and also relating it to other fields. For instance, talks were given on the following topics:

Mathematics and photography, by Robert Lockie and John Kenney
Soap films, by Lois Dodge and Betty Bushnell
Weights computing, by Irene Fisher
The planimeter, by Mary Jane Gill.

As often as possible the program chairman chooses speakers who have had interesting experiences concerning the topic. For example, Miss Fisher, who talked on the method used in finding the weight of certain parts of an airplane, had actually worked as a weights computer; and Mr. Lockie and Mr. Kenney are very proficient with the camera.

We have found that mathematics can be used for recreation as well as for practical applications. One talk that proved to be very relaxing and amusing was on the subject:

Mathematical games, by Phyllis Valentine.

Each year the Math Club invites seniors from the local high schools to a meeting in order to acquaint them with the University campus and the organization. During this meeting, the audience was surprised and delighted to hear a talk entitled:

Geometrical exercises in paper folding, by Phyllis Valentine and Mrs. Rosemary Trautman.

The Club also held two parties during the year, the annual Christmas party for the faculty and a St. Patrick's Day party, and a closing picnic sponsored by the faculty.

Elections were held at the St. Patrick's Day party, and the following people were chosen as officers: President, Irene Fisher; Vice-President, Jane Noller; Treasurer, Robert Lockie; Secretary, Ruth Cohen.

Mathematics Club, Illinois Institute of Technology

The I. T. Math Club met bi-monthly during the Fall semester, 1944, to hear the following papers:

Impossibility of trisection, by Samuel Karlin
Bernoulli's numbers, by Harley Flanders
Fractions, by Dr. L. R. Ford
Algebraic and geometric inequalities, by Dr. H. Wall
A numerical check in the solution of linear equations, by Dr. H. Rheingold
Certain Diophantine equations, by Norbert Kaufman
The Lorentz transformation, by Willard Skolnik.

Officers of the club were: President, Harley Flanders; Secretary, Norbert Kaufman; Faculty Sponsor, Dr. L. R. Ford.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

The Meaning of Relativity. By Albert Einstein. Princeton University Press, 1945. 135 pages. $2.00.

For a modern mathematician the most natural approach to a physical theory appears to be through the postulational method; we write down certain equations, develop their consequences, and after establishing a correspondence between some quantities in the equations and quantities measured by the physicist we check these consequences with the results of his experiments; if they agree we accept the theory. A mathematician may be satisfied with this situation, and for him there may be no question of further meaning. However he would not deny that there are other legitimate questions, *e.g.*, the question of how a physicist arrives at the postulates or equations which lie at the basis of the successful theory. A considerable portion of the book is devoted to discussions of this character.

The theory is developed in three stages in each of which the mathematical apparatus is introduced in close contact with physical considerations. Pre-relativity physics is first treated by means of a restricted form of three-dimensional tensor analysis, which as the author emphasizes automatically furnishes equations invariant under rotations; in the words of the author it embodies "the principle of relativity with respect to rotations." Relativity with respect to the state of motion appears in the second stage motivated by the results of the Michelson-Morley experiments; these results suggest that the laws of physics should possess invariance under Lorentz transformations—the analogues of the rotations of the first stage, and in keeping with this tensor analysis is extended by increasing the number of dimensions (and introducing imaginary components). The 1946 reader will be especially interested in the mass-energy relation which is a consequence of this development, and this interest will be enhanced by a footnote mentioning in this connection radio-active decomposition (of course, these things were first stated by Einstein back in 1905). Again in the third stage the introduction of new conceptions is done in close touch with physics; the development is suggested by the fundamental fact of equivalence between inertial and gravitational mass; the mathematical theory which embodies this suggestion is built on the model of the theory of surfaces and involves a further extension of tensor analysis and the idea of invariance. As before, the reader is not assumed to have previous familiarity with the mathe-

matical tools—they are developed—with mastery and elegance—in the body of the book. In spite of this the book should appeal not so much to a beginner as to one who has acquired the technique of calculations but has failed to develop complete inner conviction; it supplies the connection between the mathematical apparatus and the physical conceptions—in this sense it justifies its title—*The Meaning of Relativity*.

The fact that this book first published in 1922 is now reprinted without changes (except for the addition of an appendix) seems to indicate that the period of the last twenty or twenty-five years was a period of stagnation in the development of the theory which showed such an exuberant growth in the first years of its existence. This is more or less true; although the number of papers published on the theory of relativity during these years has been considerable it is difficult to name many points in which conspicuous success has been achieved; from this point of view the republication without revision may be to a great extent justified. However during this period, largely under the influence of the theory of relativity itself, our attitudes have undergone certain changes, our thinking has become so to say more radical; one may feel sorry that this attitude is not reflected to full extent in the book.

As an example we may bring up the cosmological question to which, however, much space is devoted (about one-fourth of the book, counting the appendix devoted entirely to this question and written in 1945). If the principle of relativity may be formulated by saying that no time direction is preferred as far as the *laws* of physics are concerned, it is very tempting to assume that the same idea applies to *distribution of matter*. Put another way, it seems inconsistent to assume invariance (under Lorentz transformations) of the differential equations and not to assume the same invariance for the initial conditions. Surprising as it is, the last assumption is *not* made—and it would have considerable bearing on the cosmological problem even in special relativity. Instead of uniform distribution of points in three-space we would have to consider uniform distribution of straight lines (representing matter) with respect to pseudo-euclidean metric in space time. This point of view advanced by E. A. Milne seems to be entirely neglected. Another logical consequence of the general point of view, the equivalence of all photons, or what may be called relativity of color, is also not brought out. A reader brought up on general relativity theory will also find it difficult to understand the author's point of view when he speaks of the *size* of particles.

In spite of all that has been said, and although it is not conceded in general that the best way to learn a theory is to travel the road that originally was the path of discovery, these discussions coming from the master who originated both the special and the general theory are highly interesting. It may be said that some of them permit us actually a glimpse into the ways of working of the great author.

G. Y. Rainich

Plane and Spherical Trigonometry. Second Edition. By H. A. Simmons. New York, John Wiley and Sons, 1945. 11+387 pages. $2.25.

Plane and Spherical Trigonometry, Second Edition, by H. A. Simmons, is a rewriting of *Plane Trigonometry* by Simmons and Gore. Marked innovations appear throughout the text, as to arrangement, revision of content, modification of definitions, extension of scope and the contrast of heavy type with lighter printing (to be much commended). The author prefers to begin with the general angle rather than with the positive acute angle as was done in the earlier volume.

The titles of the chapters follow: I The Trigonometric Functions of any Angle, II Solution of Right Triangles by Natural Functions, III Trigonometric Identities, IV The Reduction Formulas, V Radian and Mil Measure, VI Line Values, Graphs, and Periods of the Functions, VII The Addition Formulas and Related Topics, VIII Logarithms, IX Solution of Triangles by Logarithms, X Inverse Trigonometric Functions, XI Trigonometric Equations. These chapters constitute the treatment of Plane Trigonometry.

As a foundation for Spherical Trigonometry, the chapter on Selected Material from Solid Geometry is introduced, including just that content which should afford the reader an intelligent understanding of Spherical Trigonometry and some important applications. The material is well chosen and well arranged, with no excess impedimenta.

The remaining chapters are XIII Spherical Trigonometry, treated quite in detail, and a chapter on Applications to Navigation and Astronomy. There are included three appendices, namely A. Complex Numbers, B. The Slide Rule and C. Important Formulas.

In chapter one are found basic definitions, rectangular and polar coordinates, vectors and the six fundamental trigonometric functions plus the three special functions. The student is thus made acquainted with the above basics earlier rather than delaying some of them till later. This chapter is not too long but does give the learner an excellent foothold. In chapter two, the six general functions are specialized for solution of the right triangle, using three place tables of natural functions. Chapters three, four, five and six are treated in the traditional method, but in lucid style. These six chapters mark the text as an excellent self-teaching as well as a good classroom instrument.

Chapter seven treats of addition formulas, the half-angle in terms of cosine alone, and sine and cosine, the product formulas for sines and cosines. The half-angle formulas in terms of sides are delayed till chapter nine. Logarithms are treated traditionally, but with quickening variety in the definitions and exercises. In the chapter on solution of triangles by logarithms there is included a brief treatment of triangulation with examples.

Selections marked with the star may be omitted, but a good class should include them or nearly all. The author holds that every student should own a slide rule, and begin early to use it. The author has made a worthy contribution to the list of trigonometry texts.

RICHARD MORRIS

NEW BOOKS RECEIVED

Algebra. A Second Course. By R. O. Cornett. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+313 pages. $2.00.

Analytic Geometry. By R. R. Middlemiss. New York and London, McGraw-Hill Book Co., Inc., 1945. 7+306 pages. $2.75.

Lectures on the Theory of Functions. By J. E. Littlewood. Oxford University Press, 1944. 243 pages. $5.50.

Table of Arc Sin x. Prepared by Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1945. 19+124 pages. $3.50.

Tables of Associated Legendre Functions. Prepared by Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1945. 46+306 pages. $5.00.

The Development of Mathematics. Second Edition. By E. T. Bell. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+637 pages. $5.00.

The Social Impact of Science: A Select Bibliography with a Section on Atomic Power. By the Subcommittee on War Mobilization of the Senate Military Affairs Committee. Washington, D. C., Superintendent of Documents, 1945. 5+51 pages. $0.15.

Victoria Through the Looking-glass. The Life of Lewis Carroll. By F. B. Lennon. New York, Simon and Schuster, 1945. 15+387 pages. $3.50.

Lectures on Probability and Statistics. By E. L. Dodd. Austin, University of Texas Press, 1945. 44 pages.

Higher Algebra for Schools. By W. L. Ferrar. Oxford University Press, 1945. 8+214 pages. $3.75.

College Algebra. Second Edition. By P. K. Rees and F. W. Sparks. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+403 pages. $2.50.

The Isoperimetric Problem. By H. Schwerdtfeger. University of Adelaide, 1945. 1+14 pages.

Introduction to Non-Euclidean Geometry. By H. E. Wolfe. New York, The Dryden Press, 1945. 13+247 pages. $4.50.

The law of diminishing returns. If a female slave sixteen years of age, brings thirty-two (*nischas*), what will one aged twenty cost? Answer: 25½ *nischas*. Bhascara (12th century), *Li'la' vati'*, 76. Colebrooke's translation, 1817.—*Contributed.*

* * * *

Tell all. Most learned algebraist! tell various pairs of integer numbers, the difference of which is a square, and the sum of their squares a cube. Bhascara, *Vi'ja Gan'ita*, 182. Colebrooke's translation.—*Contributed.*

* * * *

How to shine at a faculty tea. As the sun obscures the stars, so does the proficient eclipse the glory of other astronomers in an assembly of people, by the recital of algebraic problems, and still more by their solution. Brahmegupta (sixth century), *Problems*, 102. Colebrooke's translation, 1817.—*Contributed.*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 706. *Proposed by D. H. Browne, Buffalo, N. Y.*

For what values of n does $(a+b)^n$ yield only odd coefficients?

E 707. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a quadrangle having given the lengths of the sides and of the line joining the midpoints of the diagonals.

E 708. *Proposed by M. P. de Regt, Todd Pacific Shipyards, Tacoma, Washington*

Is there a succession of digits which represents a prime in all systems of notation of radix r, $2 \leqq r \leqq 10$? If so, find the smallest one.

E 709. *Proposed by N. A. Court, University of Oklahoma*

In a given sphere to inscribe a tetrahedron so that three concurrent edges shall pass through three given points and the plane of the three remaining edges shall be parallel to the plane determined by the three given points.

E 710. *Proposed by D. H. Lehmer, Aberdeen Proving Ground, Md.*

Find the inverse of the symmetric matrix of the nth order: $A = \|a_{ij}\|$, in which $a_{ij} = i/j$ for $i \leqq j$.

SOLUTIONS

Duodenary Cyclic Numbers

E 672 [1945, 274]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the cyclic numbers in the duodenary scale. (A number is said to be *cyclic* if every cyclic permutation of its digits produces a multiple of the number.)

Solution by E. P. Starke, Rutgers University. Let S be the radix of any scale of notation, and let q be prime to S. It is known that: if S^k is the lowest power of S such that $S^k - 1$ is divisible by q, then $(S^k - 1)/q = C$ is a cyclic number; if r_i is the remainder when S^i is divided by q, $0 < i \leqq k$, then the effect of multiplying C by r_i is to transfer i digits cyclically from the left of C to the right;

if q exceeds S, C must be built up to k digits by adding zeros at the left. (See S. Guttman, *On Cyclic Numbers*, this MONTHLY [1934, 159–166]. See also R. E. Moritz, *On Products Whose Digits Are Cyclical Permutations of the Digits of the Multiplicand*, this MONTHLY [1927, 33–37].)

Conversely, let N be a number of k digits in the scale of S, and let N_m be the number obtained by transferring the first m digits of N cyclically to the right. If H_m is the number represented by the first m digits of N, then it is easy to see that

$$N_m = NS^m - H_m(S^k - 1).$$

If N_m is a cyclic number so that for all values of m it is a multiple of N, say $N_m = a_m N$, we have

$$S^m - a_m = H_m(S^k - 1)/N.$$

Thus $(S^k-1)/N$ is an integer, say q, relatively prime to S, or it is q divided by a common factor, f, of $M_1, M_2, \cdots, M_{k-1}, N$. Hence

$$N = f(S^k - 1)/q.$$

Now the conditions on f require that f be a divisor of every digit of N, and hence every digit of $(S^k-1)/q$ must be smaller than S/f. If S^k is not the smallest power of S such that S^k-1 is divisible by q, the effect is merely to take the former number C and write its digits two or more times in succession to form a single number, which is obviously cyclic if C is.

We obtain cyclic numbers, C, in the scale of twelve corresponding to each q relatively prime to twelve. There are infinitely many of them (written here in the scale of twelve with X for the digit ten and E for the digit eleven):

$$(10^1 - 1)/E = 1, \qquad (10^4 - 1)/5 = 2497, \qquad (10^6 - 1)/7 = 186X35,$$
$$(10^2 - 1)/EE = 01, \qquad (10^{14} - 1)/15 = 08579214E36429X7, \quad \textit{etc.}$$

The Proposer probably expects to exclude numbers with initial digit zero, the number 1, and also the trivial extensions like 24972497. The significant results are then 2497 and $186X35$.

Also solved by the Proposer, by using the fact that the cyclic numbers contained in the system to the base S are the periods of the fractions $1/m$, where m is a whole number less than, and prime to, S. In the duodenary system we have $m=5$, 7, and 11, the last case being trivial. For $m=5$ and 7 we find the required cyclic numbers 2497 and $186X35$ respectively.

Magic Squares That Are Zero Determinants

E 674 [1945, 274]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that a pandiagonal magic square of order 4, when regarded as a determinant, has the value zero.

Solution by E. D. Schell, Arlington, Va. Following Kraitchik's Mathematical Recreations (p. 186), we designate the general square as follows:

$$\begin{array}{cccc} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{array}$$

Let S be the common sum of each of the rows, columns, and diagonals, including the broken diagonals such as A, N, K, H. These sixteen sums yield twelve independent relations, making it possible to eliminate all but four of the unknowns. This elimination yields the following square:

$$\begin{array}{cccc} A & B & C & S-A-B-C \\ E & S-A-B-E & A-C+E & B+C-E \\ S/2-C & A+B+C-S/2 & S/2-A & S/2-B \\ S/2-A+C-E & S/2-B-C+E & S/2-E & A+B+E-S/2 \end{array}$$

Now, subtracting the first column from the third, the second column from the fourth, and factoring out $C-A$ and $S-A-2B-C$, leaves columns two and four equal. Hence the determinant of the square is zero.

Also solved by D. W. Alling and the Proposer.

Editorial Note. On p. 212 of the 11th edition of Rouse Ball's *Mathematical Recreations and Essays* we find Berghot's general form for any magic square of the fourth order, along with the conditions which render it pandiagonal. If we eliminate a, b, c by means of these conditions, and then, on the resulting array, add the fourth row to the first, the third row to the second, then subtract the first column from the third and the second column from the fourth, we obtain an array of obviously zero determinant.

The Proposer was originally seeking magic squares having zero determinants. He conjectures that a minimum necessary, though not sufficient, requirement seems to be that the square be pandiagonally magic. Of a number of trials he found no pandiagonal magic square of order five which did not have a zero determinant, nor any of order seven which did have a zero determinant. Several questions are thus raised. In particular, do all fifth order pandiagonal magic squares have zero determinants?

The First Day of a Century

E 676 [1945, 340]. *Proposed by T. R. Running, University of Michigan*

Show that in using the Gregorian calendar the first day of a century cannot occur on a Sunday, a Wednesday, or a Friday.

I. *Solution by L. S. Shively, Ball State Teachers College.* The present century, which began on Jan. 1, 1901, has in it the year 2000. Hence it contains 25 leap years, and it has 36525 days. Each of the next three centuries has 36524 days.

Jan. 1, 1901 was a Tuesday, and since $36525 \equiv 6 \pmod 7$, the last day of the present century will be a Sunday. Consequently, Jan. 1, 2001 will be a Monday.

And since $36524 \equiv 5 \pmod 7$ it follows that Jan. 1, 2101 will be a Saturday, Jan. 1, 2201 a Thursday, and Jan. 1, 2301 a Tuesday.

Each succeeding cycle of 400 years beginning with Jan. 1, 2301 will repeat the same days of the week as the 400 years beginning Jan. 1, 1901. Therefore the first day of a century cannot be a Sunday, a Wednesday, or a Friday.

In case a calendar of 1901 is not at hand to verify the fact that Jan. 1 of that year fell on Tuesday, we note that the number of days of the first 44 years of the century is $(365)(44)+11$, a number which is congruent to 6 (mod 7). Since Jan. 1, 1945 was a Monday, Jan. 1, 1901 was a Tuesday.

II. *Solution by G. W. Petrie, USNR Midshipmen's School, Notre Dame.* It is easily shown that the dth day of year Y is the Dth day of the week (starting with Sunday $=1$), where

$$D \equiv Y + \left[\frac{Y-1}{4}\right] - \left[\frac{Y-1}{100}\right] + \left[\frac{Y-1}{400}\right] + d, \text{ mod } 7,$$

where $[x]$ indicates the greatest integer in x. For the first day of a century we have

$$d = 1 \quad \text{and} \quad Y = 100C + 1.$$

Hence

$$D \equiv 2(1 - C) + \left[\frac{C}{4}\right], \text{ mod } 7.$$

D is seen to be of period 4 and to take on the values 2, 0, 5, and 3 in succession. Hence the first day of a century cannot be a Sunday, a Wednesday, or a Friday.

Also solved by Colin Blyth, Betty Boyd, D. H. Browne, Monte Dernham, F. W. Saunders, E. D. Schell, and the Proposer. Gordon Pall pointed out that this problem occurs as puzzle 416 in H. E. Dudeney's *Amusements in Mathematics.*

Editorial Note. This, and most elementary calendar problems, can be solved by use of the calendar formulae of Gauss and Zeller. For the former see *Mathematical Recreations* by Kraitchik, for the latter see *Elementary Number Theory* by Uspensky and Heaslet; both may be found in the earlier editions of *Mathematical Recreations and Essays* by Ball.

A Triakis-Tetrahedron

E 677 [1945, 340]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let the altitudes of a tetrahedron $ABCD$ meet the circumsphere again in the points A', B', C', D'. Show that the volume of the solid $ABCDA'B'C'D'$ is three times that of the given tetrahedron.

Solution by L. M. Kelly, U. S. Coast Guard Academy. Let M, G, O be the Monge point, centroid, and circumcenter of $ABCD$ and let M_a, G_a, O_a, H_a be the feet of the perpendiculars dropped from M, G, O, and A on the face BCD.

Let a perpendicular from O to AA' meet AA' in point P. Then $AP=PA'$. Also, from elementary geometry of the tetrahedron, $\frac{1}{2}(MM_a+OO_a)=GG_a=\frac{1}{4}AH_a$. Therefore

$$\begin{aligned} MM_a &= \tfrac{1}{2}AH_a - OO_a = \tfrac{1}{2}(AP + OO_a) - OO_a \\ &= \tfrac{1}{2}(AP - OO_a) = \tfrac{1}{2}(PA' - PH_a) = \tfrac{1}{2}H_aA'. \end{aligned}$$

Hence volume of $A'BCD$ is twice the volume of $MBCD$, with similar relations for the other faces. This proves the proposition.

Also solved by the Proposer.

Circular Permutations

E 678 [1945, 340]. *Proposed by H. D. Grossman, New York City*

Prove that the number of circular permutations of $n=\sum_{i=1}^{k}a_i$ objects of which a_1 are alike, a_2 are alike, *etc.*, is

$$\frac{1}{n}\sum_{d}\phi(d)\,\frac{(n/d)!}{(a_1/d)!(a_2/d)!\cdots(a_k/d)!},$$

where d ranges over all divisors of the g.c.d. of $a_1, a_2, \cdots, a_k$.

Solution by H. S. Grant, Rutgers University. A formula for the number of circular permutations P referred to in this problem was given in the solution to problem No. 519 in the April 1944 issue of the *National Mathematics Magazine*. By using the Möbius function $\mu(s)$ with its well known properties, we have, in terms of the original notation of problem No. 519,

$$P_{d'} = \sum_{s\,|\,d/d'}\mu(s)\left[\frac{\left(\sum_{i=1}^{n}(x_i/d's)\right)!}{\prod_{i=1}^{n}(x_i/d's)!}\right] \div \sum_{i=1}^{n}(x_i/d'),$$

where s ranges over the $v(d/d')$ divisors of d/d', $v(x)$ signifying the number of divisors of x. The required number of circular permutations P is then given by

$$P = \sum_{d'\,|\,d} P_{d'},$$

where d' ranges over the $v(d)$ divisors of d.

In the notation of Grossman's proposal we have

$$P_{d'} = \sum_{s\,|\,D/d'}\mu(s)\left[\frac{\left(\sum_{i=1}^{k}(a_i/d's)\right)!}{\prod_{i=1}^{k}(a_i/d's)!}\right] \div \sum_{i=1}^{k}(a_i/d'),$$

where we have written $D=(a_1, a_2, \cdots, a_k)$, and s ranges over the $v(D/d')$ divisors of D/d'. Since $\sum_{i=1}^{k}a_i=n$, this readily reduces to

$$P_{d'} = \frac{1}{n} \sum_{s|D/d'} d'\mu(s) \left[\frac{(n/d's)!}{\prod_{i=1}^{k} (a_i/d's)!} \right],$$

whence

$$P = \sum_{d'|D} \left\{ \frac{1}{n} \sum_{s|D/d'} d'\mu(s) \left[\frac{(n/d's)!}{\prod_{i=1}^{k} (a_i/d's)!} \right] \right\}.$$

For each of the $v(D)$ values of d' there are $v(D/d')$ values of s, so that the double summation consists of a totality of $\sum_{d'|D} v(D/d')$ terms. Because of the symmetry in the values assumed by d' and s we have

$$P = \frac{1}{n} \sum_{d's|D} \sum d'\mu(s) \left[\frac{(n/d's)!}{\prod_{i=1}^{k} (a_i/d's)!} \right]$$

summed over the $\sum_{d'|D} v(D/d') = \sum_{d'|D} v(d')$ pairs of values of d' and s for which $d's$ divides D. These pairs of values of d' and s may be obtained by securing the $\sum_{d|D} v(d)$ pairs of values satisfying $d's = d$, d any divisor of D. Whence

$$P = \frac{1}{n} \sum_{d|D} \sum_{d's=d} d'\mu(s) \left[\frac{(n/d)!}{\prod_{i=1}^{k} (a_i/d)!} \right].$$

Since $\sum_{d's=d} d'\mu(s) = \phi(d)$ (Hardy and Wright, *Theory of Numbers*, p. 234),

$$P = \frac{1}{n} \sum_{d|D} \phi(d) \left[\frac{(n/d)!}{\prod_{i=1}^{k} (a_i/d)!} \right].$$

A Spherical Helix

E 679 [1945, 341]. *Proposed by Marshall Naul, Cumberland, Md.*

Compute the length of the curve of intersection of the unit sphere around the origin with the right helicoid $z = \theta$.

Editorial Note. We readily find the following parametric representation for the curve of intersection,

$$x = (1 - \theta^2)^{1/2} \sin \theta, \qquad y = (1 - \theta^2)^{1/2} \cos \theta, \qquad z = \theta.$$

Then, letting s be the required arc length and letting primes indicate differentiation with respect to θ, we have

$$s = 4 \int_0^1 (x'^2 + y'^2 + z'^2)^{1/2} d\theta = 4 \int_0^1 \left[\frac{\theta^4 - 2\theta^2 + 2}{1 - \theta^2} \right]^{1/2} d\theta.$$

The numerical value of the integral may be approximated by several methods.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4189. *Proposed by Albert Wilansky, Brown University*

Prove that

$$\sum_{r=0}^{m}(-1)^r\binom{m+k+\alpha}{r+k+\alpha}\binom{r+k}{k}=\binom{m+\alpha-1}{m}, \qquad \alpha \geqq 0.$$

4190. *Proposed by Norman Anning, University of Michigan*

If a, b, c, R are integers such that $a^2+b^2+c^2=R^2$, solve in integers the simultaneous equations

$$x^2+y^2+z^2=R^2$$
$$ax+by+cz=0.$$

There are at least four solutions.

4191. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\frac{1}{\cosh(\pi/2)}-\frac{1}{3\cosh(3\pi/2)}+\frac{1}{5\cosh(5\pi/2)}-\cdots=\frac{\pi}{8}.$$

4192. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron $A_1A_2A_3A_4$, if P is the centroid of its antipedal tetrahedron $ABCD$, its barycentric coordinates are inversely proportional to the squares of its distances to A_1, A_2, A_3, A_4; and conversely.

4169 [1945, 400]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

The tangents at the vertices A, B, C of a given triangle to its Feuerbach hyperbola form a triangle whose conjugate circle is tangent to the ninepoint circle of ABC at its Feuerbach point.

SOLUTIONS

A Trigonometric Equation

4136 [1944, 533]. *Proposed by H. S. M. Coxeter, University of Toronto*

The equation

$$\cos \pi x + \cos \pi y + \cos \pi z = 0, \qquad 0 \leqq x \leqq \tfrac{1}{2} \leqq y \leqq z \leqq 1,$$

has the trivial solutions $y=\frac{1}{2}$, $z=1-x$, and $y=2/3-x$, $z=2/3+x$. It has also the non-trivial solution $x=1/5$, $y=3/5$, $z=2/3$. Prove that it has no other rational solutions.

Solution by W. J. R. Crosby, Saskatoon, Can. It will be shown* that the only solutions of the equation

$$\cos 2\pi r_1 + \cos 2\pi r_2 + \cos 2\pi r_3 = 0,$$

where r_1, r_2, r_3 are non-negative rational numbers less than unity, are of one of the following types:

(a) $$\cos \tfrac{1}{2}\pi + \cos 2\pi r + \cos 2\pi(\pm \tfrac{1}{2} \pm r) = 0,$$

(b) $$\cos 2\pi r + \cos 2\pi(r + \tfrac{1}{3}) + \cos 2\pi(r + \tfrac{2}{3}) = 0,$$

(c) $$\cos (2\pi/3) + \cos (\pi/5) + \cos (3\pi/5) = 0,$$

and equivalent solutions obtained from these by means of the relations

$$\cos 2\pi r = \cos 2\pi(1 - r) \quad \text{and} \quad -\cos 2\pi r = \cos 2\pi(\pm \tfrac{1}{2} \pm r),$$

where r is a non-negative rational less than unity. This will certainly establish the statement made in the problem.

Let $r_k = n_k/d_k$, where $0 \leqq n < d$ and n_k, d_k are mutually prime if $n_k \neq 0$. Let p be the largest prime which is a divisor of d_1, d_2, or d_3. Then we can find numbers δ_k, l_k, c_k, ν_k such that

$$d = \delta_k p^{l_k}, \quad \text{and} \quad n_k = c_k\delta_k + \nu_k p^{l_k},$$

where δ_k is prime to p, $0 \leqq c < p^{l_k}$, $c_k = 0$ if $l_k = 0$ but otherwise c_k is prime to p. Therefore, if we write f_k for ν_k/δ_k, we shall have

$$r_k = n_k/d_k = \nu_k/\delta_k + c_k/p^{l_k} = f_k + c/p^{l_k}.$$

We shall assume that $l_1 \geqq l_2 \geqq l_3$.
Now let

$$g_k(x) = \tfrac{1}{2}(e^{2\pi i f_k} x^{c_k p^{l_1 - l_k}} + e^{-2\pi i f_k} x^{p^{l_1}} - c_k p^{l_1 - l_k} \quad \text{if} \quad c_k \neq 0,$$
$$= \cos 2\pi r_k \quad \text{if} \quad c_k = 0,$$

and let

$$U(x) = \sum_{k=1}^{3} g_k(x).$$

Then

$$g_k(e^{2\pi i/p^{l_1}}) = \cos 2\pi r_k,$$

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* The method is an adaptation of a method employed to solve a similar problem by P. Gordan: Math. Ann., Vol. 12, 1877, p. 29.

and consequently, if r_1, r_2, r_3 satisfy the given equation,

$$U(e^{(2\pi i/p^{l_1})}) = 0.$$

L. Kronecker* has proved that the polynomial

$$P(x) = 1 + x^{p^{l_1-1}} + x^{2p^{l_1-1}} + \cdots + x^{(p-1)p^{l_1-1}}$$

is not the product of two polynomials of smaller degree of which the coefficients are rational functions of sth roots of unity, if s is not a multiple of p. From this result we first deduce two lemmas.

LEMMA 1. *The polynomial $U(x)$ is divisible by the polynomial $P(x)$.*

Proof: The polynomials $U(x)$ and $P(x)$ are not prime to each other since both vanish for $x=\exp(2\pi i/p^{l_1})$. The greatest common divisor of these polynomials has coefficients which are rational functions of the coefficients of $U(x)$ and $P(x)$. Therefore, by Kronecker's theorem, the greatest common divisor must be $P(x)$.

LEMMA 2. *If we express $U(x)$ as a sum of polynomials $\sum_t U_t(x)$, where $U_t(x)$ contains those terms of $U(x)$ of the form bx^c with $c\equiv t$ modulo p^{l_1-1}, then $U(x)$ is divisible by $P(x)$, and consequently $U(\exp(2\pi i/p^{l_1}))=0$.*

Proof: If we divide $U_t(x)$ by $P(x)$, we obtain a relation of the form $U_t(x)=P(x)Q_t(x)+R_t(x)$, where the degree of $R_t(x)$ is less than that of $P(x)$, and the index of every power of x in $Q_t(x)$ and $R_t(x)$ is congruent to t modulo p^{l_1-1}. From this we deduce that

$$U(x) = \sum_t U_t(x) = P(x)\sum_t Q_t(x) + \sum_t R_t(x),$$

and therefore that $\sum_t R_t(x)=0$, by *Lemma* 1. Consequently $R_t(x)=0$ for each t.

We now divide our discussion of possible solutions of the equation into a number of special cases.

1. $l_1=1\geqq l_2\geqq l_3$. By *Lemma* 1, since the degree of $U(x)$ is less than p, and the degree of $P(x)$ is $p-1$, we must have $U(x)=mP(x)$ where m is a constant.

1.1 If $l_1=l_2=l_3=1$, then $U(0)=0$, and consequently $m=0$ since $P(0)=1$. Therefore $U(x)$ is zero for every x. If any one of the functions g_1, g_2, g_3, vanishes identically, it must contain only one power of x. If the sum of two terms representing two of these functions vanishes, it is easily seen that the sum of the two functions is zero, and therefore that the remaining function vanishes identically. Therefore either one of the functions g_1, g_2, g_3 contains only one power of x, or else the three functions each have the same two powers of x.

1.11 If one of the functions contains only one power of x, then for the corresponding value of k we find that $c_k=p-c_k$, and therefore $2c_k=p$, and we conclude that $p=2$ and $c_k=1$. Then, since p is the largest prime in d_1, d_2, d_3, we

* L. Kronecker: Journal de Math., Vol. 19, 1854, p. 177–192.

must have $d_1=d_2=d_3=2$. But this provides no solution for the problem and is therefore of no interest.

1.12. If the three functions g_1, g_2, g_3 each contain the same two powers of x, let b_1, b_2, b_3 be the three coefficients of one of the powers of x. Then the three coefficients of the other power of x are $1/b_1$, $1/b_2$, $1/b_3$, and $b_1+b_2+b_3=0$, $1/b_1+1/b_2+1/b_3=0$, *i.e.*, $b_2b_3+b_3b_1+b_1b_2=0$. Therefore b_1, b_2, b_3 are the three roots of the equation

$$z^3 - e^{2\pi i(3s)} = 0, \quad \text{where} \quad b_1b_2b_3 = e^{2\pi i(3s)}.$$

Consequently we may take $b_1=\exp(2\pi is)$, $b_2=\exp(2\pi i(s+1/3))$, $b_3=\exp(2\pi i(s+2/3))$. The corresponding solution is found to be

$$\cos 2\pi(s + c/p) + \cos 2\pi(s + 1/3 + c/p) + \cos 2\pi(s + 2/3 + c/p) = 0,$$

which is of type (b).

1.2. If $l_1=l_2=1$, $l_3=0$, then $m=\cos 2\pi r_3$.

1.21. If further $\cos 2\pi r_3=0$, then $\cos 2\pi r_1+\cos 2\pi r_2=0$, and consequently $r_2=\pm\frac{1}{2}\pm r_1$. We thus obtain a solution which is evidently of type (a).

1.22. If $\cos 2\pi r_3\neq 0$, then $U(x)=P(x)\cos 2\pi r_3$. Now the number of different powers of x in $P(x)$ is p, and in $U(x)$, 2, 3 or 5. Hence p is one of the numbers 2, 3, 5. But if $p=2$, the two functions g_1, g_2 contain only one power of x, and this case has been treated before in 1.11.

1.221. If $p=3$, then comparing coefficients we find that

$$\tfrac{1}{2}e^{2\pi if_1} + \tfrac{1}{2}e^{2\pi if_2} = \tfrac{1}{2}e^{-2\pi if_1} + \tfrac{1}{2}e^{-2\pi if_2} = \cos 2\pi r_3,$$

or

$$\tfrac{1}{2}e^{2\pi if_1} + \tfrac{1}{2}e^{-2\pi if_2} = \tfrac{1}{2}e^{-2\pi if_1} + \tfrac{1}{2}e^{2\pi if_2} = \cos 2\pi r_3.$$

In the first case $f_1=-f_2=\pm r_3$, and in the second $f_1=f_2=\pm r_3$. The corresponding solution in each case is of type (b).

1.222. If $p=5$, then

$$\tfrac{1}{2}e^{2\pi if_1} = \tfrac{1}{2}e^{-2\pi if_1} = \tfrac{1}{2}e^{2\pi if_2} = \tfrac{1}{2}e^{2\pi if_2} = \cos 2\pi r.$$

Hence $f_1=f_2=0$ and $\cos 2\pi r_3=\frac{1}{2}$, *i.e.* $r_3=1/6$, or $f_1=f_2=\frac{1}{2}$ and $\cos 2\pi r_3=-\frac{1}{2}$, *i.e.* $r_3=1/3$. If now c_1, c_2, c_3 are chosen in such a way that $U(x)$ has five different powers of x, solutions are obtained which are of type (c).

1.3. If $l_1=1$, $l_2=l_3=0$, then

$$U(x) = (\cos 2\pi r_2 + \cos 2\pi r_3)P(x).$$

1.31. If further $\cos 2\pi r_2+\cos 2\pi r_3=0$, then $\cos 2\pi r_1=0$, and the solution must be of type (a).

1.32. If $\cos 2\pi r_2+\cos 2\pi r_3\neq 0$, then, since $U(x)$ has two or three different powers of x, $p=2$ or 3. But since the case $p=2$ has been considered in 1.11, we assume that $p=3$. Then

$$\tfrac{1}{2}e^{2\pi i f_1} = \tfrac{1}{2}e^{-2\pi i f_1} = \cos 2\pi r_2 + \cos 2\pi r_3,$$

and each of these quantities must evidently be equal to $\pm\frac{1}{2}$. Hence $f_1=0$ and $\cos 2\pi r_2+\cos 2\pi r_3=\frac{1}{2}$, or $f_1=\frac{1}{2}$ and $\cos 2\pi r_2+\cos 2\pi r_3=-\frac{1}{2}$, where the denominators of r_2 and r_3 have no prime factor greater than 2. But this is obviously impossible.

2. Let us next assume that $l_1\geqq 2$, $l_1\geqq l_2\geqq l_3$. The indices of the powers of x appearing in g_1, g_2, g_3 are respectively c_1, $p^{l_1}-c$; $c_2p^{l_1-l_2}$, $p^{l_1}-c_2p^{l_1-l_2}$; $c_3p^{l_1-l_3}$, $p^{l_1}-cp^{l_1-l_3}$. When two of these indices are congruent modulo p^{l_1-1}, we shall say that the two corresponding terms of $U(x)$ are similar. Our conclusions in each case will be derived from *Lemma* 2.

2.1. If $l_1>l_2\geqq l_3$, the terms of g_1 are dissimilar to the terms of g_2 and g_3 and are similar to each other only if $p=l_1=2$ and $c_1=1$.

2.11. If $p=l_1=2$, then $g_1(\exp(2\pi i/4))=\cos 2\pi r_1=0$. We thus obtain a solution of type (a).

2.12. If $p\neq 2$, then $P(x)$ divides each term of $g_1(x)$. This is evidently impossible.

2.2. If $l_1=l_2\geqq l_3$, then the terms of g_3 are dissimilar to the terms of g_1 and g_2, and similar to each other only if $l_3=0$ or 1, or if $l_3=p=2$.

2.21. If $l_3=0$, or if $l_3=p=2$, then $\cos 2\pi r_3=0$, and we obtain a solution of type (a).

2.23. If $l_3=1$, then again $\cos 2\pi r_3=0$, and consequently $p=2$ and $l_3=2$. This is a contradiction.

2.24. If the two terms of g_3 are dissimilar to each other, then $P(x)$ divides both these terms; this is impossible.

2.3. If $l_1=l_2=l_3$, the two terms of each function g_1, g_2, g_3 are dissimilar to each other unless $p=l_1=l_2=l_3=2$, and therefore, except in this special case, there are at most three similar terms in $U(x)$.

2.31. If $p=l_1=l_2=l_3=2$, then the corresponding solution is of type (a).

2.32. If there are two pairs of three similar terms in $U(x)$, then the sum of each group of terms must vanish for $x=\exp(2\pi i/p^{l_1})$. The two equations obtained in this way have exactly the same form as the two equations for b_1, b_2, b_3 in 1.12. The corresponding solution is therefore of type (b).

Derangements

4146 [1945, 47]. *Proposed by Gaines N. Lang, Emory University*

Suppose that n men check their hats, and then each man takes a hat at random. Let $g(n)$ denote the probability that no one gets his own hat. Find the value of $g(n)$ and its limit for $n\to\infty$.

Solution by G. B. Huff, Southern Methodist University. This is the famous example of De Montmort, worked on page 24 of Coolidge's *Probability*. A reference is given there to *Essai d'analyse sur les jeux de hasard*, 2nd ed. Paris, 1713, p. 132. The following solution may be of interest.

The number of satisfactory arrangements $a(n)$ is subject to the recursion relation

$$a(n) = (n-1)[a(n-2) + a(n-1)]. \tag{1}$$

Indeed, the man numbered n must have the hat of man i, $1 \leqq i \leqq n-1$, and the latter does or does not have the hat numbered n. The first occurs in $a(n-2)$ ways and the second in $a(n-1)$ ways.

Now $g(n) = a(n)/n!$ and this with (1) yields the relation

$$g(n) - g(n-1) = -[g(n-1) - g(n-2)]/n. \tag{2}$$

Since $g(2) - g(1) = 1/2!$, it follows that $g(n) - g(n-1) = (-1)^n/n!$, and

$$g(n) = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}. \tag{3}$$

Clearly,

$$\lim_{n \to \infty} g(n) = e^{-1}.$$

Solved also by D. W. Alling, Murray Barbour, Colin Blyth, D. H. Browne, Howard Eves, R. E. Greenwood, C. O. Oakley, G. W. Petrie, Joseph Rosenbaum, Henry Scheffé, E. D. Schell, E. P. Starke, Alan Wayne, and the proposer.

Editorial Note. In addition to the above the following other references were given

(a) T. W. Anderson, Jr., *Annals of Math. Stat.* vol. 14, 1943, pp. 426 *etc.*
(b) Ball and Coxeter, *Math. Recreations and Essays*, 11th ed., p. 46.
(c) Murray Barbour, *School Science and Math.*, Prob. 883, 1925, p. 869; Prob. 938, 1927, p. 92.
(d) Chrystal, *Algebra*, vol. 2, pp. 22–25.
(e) Hall and Knight, *Higher Algebra*, 1942 ed., p. 404, ex. 3.
(f) Irving Kaplansky, *On a generalization of the Probleme des Rencontres*, this MONTHLY, 1939, 159.
(g) Levy and Roth, *Elements of Probability*, p. 97.
(h) Todhunter, *A History of the Mathematical Theory of Probability*, chap. 8, p. 93.
(i) J. V. Uspensky, *Introduction to Mathematical Probability*, pp. 168.
Elementary Number Theory, 1939, pp. 105–107.
(j) Whitworth, *Choice and Chance*, chap. 2.

Several solutions are similar to the above and the remaining solutions are similar to special cases of the applications below of two types of inversion which will be stated. Starke is the only solver who mentions one of the general inversion formulas, and one of his solutions is based on the formula in the second reference of (i). The inversion formulas lead to a known generalization of the problem. One set of the inversion formulas follows.

Given the function $\phi(r-i)$ where i, r are integers, $0 \leqq i \leqq r$, set

$$\psi(r) = \sum_{i=0}^{r} {}_rC_i\phi(r-i). \tag{1}$$

Then

$$\phi(r) = \sum_{i=0}^{r} {}_rC_i(-1)^i\psi(r-i), \tag{2}$$

and conversely.

There are several proofs. It will be shown that (1) follows from (2). The right member of (1) becomes

$$\sum_{i=0}^{r} {}_rC_i \sum_{j=0}^{r-i} {}_{r-i}C_j(-1)^j\psi(r-i-j) = \sum_{i=0}^{r} {}_rC_i \sum_{t=0}^{r-i} {}_{r-i}C_i(-1)^{r-i-t}\psi(t)$$

$$= \sum_{t=0}^{r} {}_rC_i(-1)^{r-t}\psi(t) \sum_{i=0}^{r-t} {}_{r-t}C_i(-1)^i.$$

For $t<r$ the second sum in the last expression is $(1-1)^{r-t}=0$; for $t=r$ it is unity; and this completes the proof of this part. The proof that (2) follows from (1) is similar; or we may use the fact that the determinant for the set of equations (1) is not zero.

A second proof uses the operator $\delta_r=1-U_r^{-1}$ as follows;

$$\delta_r\psi(r) = \phi(r) + \sum_{j=1}^{r-1} ({}_rC_j - {}_{r-1}C_{j-1})\phi(r-j) = \sum_{j=0}^{r-1} {}_{r-1}C_j\phi(r-j).$$

Hence

$$\delta_r^r\psi(r) = \phi(r) = (1-U_r^{-1})^r\psi(r) = \sum_{i=0}^{r} {}_rC_i(-1)^i\psi(r-i).$$

For the converse use $\bar{\delta}_r=1+U_r^{-1}$.

Another form of this inversion is

$$F(n,r) = \sum_{i=0}^{r} {}_rC_i f(n-i, r-i), \tag{3}$$

$$f(n,r) = \sum_{i=0}^{r} {}_rC_i(-1)^i F(n-i, r-i), \tag{4}$$

where i, r, n are integrs $0\leqq i\leqq r\leqq n$.

An application. Consider the permutations, P's, of the elements $1, 2, \cdots, n$ in this order, and select a particular set of r of these elements, say the first r. A particular subset of i of these r elements is chosen, and denote by $\phi(r-i)$ the number of P's for which each of the i elements is in its original position and no one of the remaining $r-i$ elements is in its original place. Then (1) gives $\psi(r)=n!$ and $\psi(r-i)=(n-i)!$. We then have from (2) the number of P's, $\phi(r)$, for which no one of the chosen r elements is in its original place

$$\phi(r) = \sum_{i=0}^{r} {}_rC_i(-1)^i(n-i)!. \tag{5}$$

In the problem $r=n$ and we have

$$\phi(n) = n!\sum_{i=0}^{n} \frac{(-1)^i}{i!} = s(n), \qquad s(0) = 1, \qquad s(1) = 0, \tag{6}$$

$$s(n) = ns(n-1) + (-1)^n.$$

Some writers call the function $s(n)$ "subfactorial n."

In the second inversion formulas we consider N objects and n properties numbered $1, 2, \cdots, n$. A set of r of these properties is arbitrarily chosen, say the first r. Denote by $M^{(r)}_{a_1a_2\ldots a_i}$ the number of objects each having the indicated i properties selected from the chosen r and none of the remaining $r-i$; for $i=0$ we write $M^{(r)}$ which means the number of objects having none of the r properties. Denote by $N^{(r)}_{a_1a_2\ldots a_i}$ the number of objects with the i indicated properties chosen from the r and none or one or more of the remaining $r-i$; $M^{(r)}_{12\ldots r} = N^{(r)}_{12\ldots r}$, and for $i=0$ we write $N^{(r)}=N$. Then obviously

$$N = \sum_{i=0}^{r} \sum M^{(r)}_{a_1a_2\ldots a_i}, \tag{7}$$

where the second sum has ${}_rC_i$ terms. Then the previous inversion suggests that

$$M^{(r)} = \sum_{i=0}^{r} (-1)^i \sum N^{(r)}_{a_1a_2\ldots a_i}. \tag{8}$$

This is also suggested by the first two or three cases $r=1, 2, 3$, which are easily computed in turn. The simplest proof seems to be the following. Let a particular object have the m properties $a_1, a_2, \cdots, a_m$ chosen from the r and also the properties $b_1, b_2, \cdots, b_k$ chosen from the $n-r$, *i.e.*, excluding the r set, and no other properties, $0<m\leqq r$, and $0\leqq k\leqq n-r$. It will be shown that this object is eliminated in the counting in the right member of (8), and this will prove (8). For $i=0$ this object is counted once; for $i>m$ it is not considered; and for $i\leqq m$ it is counted $(-1)^i\,{}_mC_i$ times. In all it is counted a number of times equal to

$$\sum_{i=0}^{m} {}_mC_i(-1)^i = (1-1)^m = 0;$$

and this completes the proof.

An application. For the N objects we take the $n!$ permutations, P's, of the elements $1, 2, \cdots, n$ in this order. The property i is that the element i is in its original place, and we select r of the elements, say the first r. Then $M^{(r)}$ denotes the number of P's for which no one of the chosen r is in its original place; $N^{(r)}_{a_1a_2\ldots a_i}=(n-i)!$; and (8) becomes the formula (5) where $\phi(r)$ is replaced by $M^{(r)}$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

The University of Wisconsin in Milwaukee wishes to exchange (or sell) some duplicate volumes of mathematical books and American and foreign periodicals. Any library or individual interested should write Professor Morris Marden, 623 W. State Street, Milwaukee, Wisconsin.

Dr. J. J. Barron of Brown University has been appointed to an assistant professorship at Marquette University.

Associate Dean Walter Bartky of the University of Chicago has been made dean.

Dr. Lipman Bers has been appointed to an assistant professorship at Syracuse University.

Associate Professors D. G. Bourgin and J. L. Doob of the University of Illinois have been promoted to professorships.

Professor Emeritus L. L. Dines of the Carnegie Institute of Technology has been appointed to a visiting professorship at the University of Saskatchewan.

Assistant Professor Churchill Eisenhart of the University of Wisconsin has been promoted to an associate professorship.

Mary L. Elveback has been appointed to an assistant professorship at Rockford College.

Dr. P. E. Guenther of the Case School of Applied Science has been promoted to an assistant professorship.

Assistant Professor Israel Halperin of Queen's University, Kingston, Ontario, has been promoted to an associate professorship.

Dr. E. D. Hellinger of Northwestern University has been appointed to a professorship.

Dr. H. A. Jordan of Harvard University has been appointed to an associate professorship at Mount Holyoke College.

Associate Professor S. C. Kleene of Amherst College has been appointed to an associate professorship at the University of Wisconsin.

Dr. C. C. Lin of the California Institute of Technology has been appointed to an assistant professorship at Brown University.

Dr. H. B. Mann of Brown University has been appointed to an associate professorship at Ohio State University.

Dr. Margaret E. Mauch of Michigan State College has been appointed to an assistant professorship at the University of Akron.

Dr. Josephine M. Mitchell of Connecticut College has been appointed to an associate professorship at Winthrop College.

Dr. R. S. Pate has been appointed to an associate professorship at the University of South Carolina.

Assistant Professor R. F. Rinehart of the Case School of Applied Science has been promoted to an associate professorship.

Assistant Professor O. F. G. Schilling of the University of Chicago has been promoted to an associate professorship.

Dr. Seymour Sherman has been appointed to an assistant professorship at the College of the University of Chicago.

Dr. C. L. Siegel has been appointed to a professorship at the Institute for Advanced Study.

Dr. Ernst Snapper of Princeton University has been appointed to an assistant professorship at the University of Southern California.

Assistant Professor N. E. Steenrod of the University of Michigan has been promoted to an associate professorship.

Dr. W. R. Transue has been appointed to an associate professorship at Kenyon College.

Dr. Bernard Vinograde of Tulane University has been appointed to an assistant professorship at Iowa State College.

Dr. E. L. Welker of the University of Illinois has been promoted to an assistant professorship.

Dr. Jacob Wolfowitz has been appointed to an associate professorship at the University of North Carolina.

Professor Emeritus H. F. Blichfeldt of Stanford University died November 16, 1945.

Assistant Professor Annie W. Fleming of Iowa State College died September 19, 1945.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

NEW SELECTIVE SERVICE POLICY PERTAINING TO SCIENTISTS

The National Headquarters of Selective Service issued Local Board Memorandum No. 115-M on November 26, 1945. Because it is of importance to mathematicians it is reproduced below.

PART I

1. *General.*—The demands of the armed forces and industry during the emergency have resulted in a curtailment in advanced studies for men having high technical and scientific qualifications. Since the cessation of active fighting, the demands of long range national interest require a resumption of these studies in order to fully develop the technical and scientific skills which have been acquired and to provide adequate teaching facilities for returning veterans who desire to resume their studies in these fields.

2. *Reconversion Working Committee on Deferment and Selective Release.*—(a) Pursuant to a request from the Director of War Mobilization and Reconversion, the Reconversion Working Committee on Deferment and Selective Release has been established to assist in the accomplishment of this purpose. The Committee is composed of representatives of the (1) Office of Scientific Research and Development; (2) Civilian Production Administration; (3) Office of Rubber Reserve; (4) Petroleum Administration for War; (5) War Department; (6) Navy Department; (7) United States Employment Service; (8) Office of Education; (9) National Roster of Scientific and Specialized Personnel; and (10) the Selective Service System. The Chairman of the Committee will be the Director of War Mobilization and Reconversion, or a staff member designated by him. The functions of the Committee are to:

(1) Indicate to the Director of War Mobilization and Reconversion the specific occupations in which shortages of personnel threaten to interfere with the national health, safety, or interest.
(2) Formulate the specific standards indicating that a man is qualified to engage in a selected occupation.
(3) Certify to the Director of the Selective Service System those individuals meeting the standards established by the Committee.
(4) Indicate to the War and Navy Departments the categories of occupations in which shortages detrimental to the national interest could be relieved by release of men from the armed forces.

PART II

1. *Operation of plan.*—The Director of the Office of War Mobilization and Reconversion will examine all proposals by the Committee relating to deferment and release, and shall transmit to the Selective Service System and the War and Navy Departments all approved proposals. The Director will transmit the recommendations for deferment to local boards through the appropriate State Directors.

2. *Classification policies.*—Pursuant to the provisions of this memorandum and under the general authority contained in paragraph 5 of Part II of Local Board Memorandum No. 115, as amended, local boards will give serious consideration to the occupational deferment of registrants engaged in the technical and scientific fields set forth in Part III of this memorandum.

PART III

1. *Advanced studies in the physical sciences or engineering.*—(a) Any registrant who is accepted by an accredited college or university as a candidate for a Master's or Doctor's degree in the physical sciences or engineering may be certified by the Office of War Mobilization and Reconversion to the Director of Selective Service as essential to the national interest in a civilian capacity.

(b) The fact that a candidate for a Master's or a Doctor's degree may engage in part-time employment or other activities will not affect his certification under this paragraph so long as his academic standing is satisfactory.

2. *University teaching in the physical sciences or engineering.*—Any registrant who is to be employed by an accredited college or university as a teacher of physical sciences or engineering may be certified by the Office of War Mobilization and Reconversion to the Director of Selective Service as essential to the national interest in a civilian capacity.

3. *University research in the physical sciences or engineering.*—(a) Any registrant (1) who is to be employed by or attached to the staff of an accredited college or university for research in the physical sciences or engineering and (2) who signifies his intention to engage in such an activity may be certified to the Director of Selective Service as essential to the national interest in a civilian capacity.

(b) Such a registrant will be certified only if (1) the research to be undertaken by the registrant contributes significantly to the national interest, and (2) inability of the individual registrant to undertake the research will result in its delay.

4. *Submission of information in certain cases.*—Any registrant who wishes to be certified under the provisions of paragraphs 1, 2, and 3 above, must present to the Office of War Mobilization and Reconversion, Washington, D. C., the following documents in triplicate:

(a) A notarized statement of his intention:

(1) To undertake graduate studies leading to a Master's or Doctor's degree in the physical sciences or engineering; or

(2) To engage in the teaching of physical sciences or engineering; or

(3) To undertake advanced research in the physical sciences or engineering on the staff of or attached to a college or university.

(b) A statement from an accredited college or university signed by a responsible official of the college or university, indicating that the registrant:

(1) Has been accepted as a candidate for a Master's or Doctor's degree in the physical sciences or engineering; or

(2) Has been accepted as a teacher of physical sciences or engineering; or

(3) Is to be employed by or attached to the staff of the college or university for research in the physical sciences or engineering, together with a statement indicating in detail the nature of the research to be performed by the registrant, the scope of the registrant's responsibilities for the research, and the necessity for securing the individual registrant for the work.

(c) A statement showing the registrant's name, address, age, Selective Service local board number and address, classification, and educational qualifications.

5. *Study leading to a B.S. or B.A. degree in physical sciences or engineering.*— (a) Any registrant who has satisfactorily completed at least three years of work leading to a Bachelor's degree in the physical sciences or engineering may be certified by the Office of War Mobilization and Reconversion to the Director of Selective Service as essential to the national interest in a civilian capacity, provided such registrant has served for a period of not less than two years in a project directly connected with the war effort.

(b) Any registrant who wishes to be certified under the provisions of this paragraph must present to the Office of War Mobilization and Reconversion, Washington, D. C., the following documents in triplicate:

(1) A notarized statement of his intention to continue undergraduate studies leading to a B.S. or B.A. degree in the physical sciences or engineering.

(2) A statement from an accredited college or university, signed by a responsible official of the college or university, indicating that the registrant has been accepted for the fourth year of study in a course leading to a B.S. or B.A. degree in the physical sciences or engineering.

(3) A statement showing the registrant's name, address, age, Selective Service local board number and address, classification, and educational qualifications.

(4) A statement from the registrant's employer that he has been engaged for at least two years in scientific war work.

COLLEGE MATHEMATICS CREDIT AND VOCATIONAL ADVICE FOR THE VETERAN

C. C. HURD, Allegheny College*

1. Introduction. Veterans are returning to college in increasing numbers. Departments of Mathematics will therefore undoubtedly wish to devote careful attention to the administrative problems of: (a) Granting appropriate credit for any in-service mathematical experience which the veteran may have had; (b) Disseminating educational and vocational advice to those veterans who may be interested in mathematics as a career.

These problems are here stated and discussed in terms of the veteran. However, it is clear that certain analogous problems exist with respect to the college student who is not a veteran.

2. The Granting of Credit. Many institutions and professional associations have accepted in principle the policy urged by the American Council on Education[1] that credit for military experience be granted "in terms of demonstrated educational competence." In order that this policy may be implemented with respect to granting credit in mathematics the Subcommittee on Examinations, War Policy Committee of the A.M.S. and M.A.A., has prepared USAFI multiple choice type subject examinations in College Algebra, Plane Trigonometry, Analytical Geometry, and Calculus.[2] The Cooperative Test Service is standardizing these examinations and the Subcommittee has stated its expectancy that various schools will desire to determine supplementary data on Form B of these tests for their own information and use. These examinations, whose topic content was determined "on results of extensive sampling of the opinions of college departments of mathematics," and which were prepared by men competent in mathematics and skilled in administration, appear to be unusually good multiple choice type examinations.

Two related questions immediately arise however. Is any multiple choice type examination, however good, adequate for the evaluation problem at hand? How shall the examination results be used with respect to the individual veteran?

Veterans will have obtained in-service mathematical experience through three principal channels: In-Service Training Schools,[3] USAFI Correspondence Courses,[4] and Off-Duty Educational Programs.[5] The questions raised above with

* Sections 1 and 2 were written while the author was on active duty at the United States Coast Guard Academy.

[1] Sound Educational Credit for Military Experience, American Council on Education, Washington, D. C., 1943, p. 2.

[2] See Activities of the Committee on Examinations this MONTHLY, March, 1945, pp. 173–175.

[3] See A Guide to the Evaluation of Educational Experiences in the Armed Services, American Council on Education, Washington, D. C., 1944.

[4] See USAFI courses in college mathematics, this MONTHLY, May, 1945, pp. 287–289.

[5] See McGrath, E. J., Off-duty education programs of the U. S. Navy, Educational Record, January, 1944, pp. 35–47.

respect to the adequacy of and the manner of using the USAFI examinations for evaluation will be discussed for:

a. The veteran who will study no mathematics in college and offers in-service mathematical experience for "terminal" or non-prerequisite credit.

b. The veteran who offers in-service mathematical experience for "pre-requisite credit" for more advanced mathematics courses and who is not a prospective mathematics major.

c. The veteran who expects to major in mathematics.

a. *Terminal Credit.* For the purpose at hand the principal advantages of using multiple choice examinations appear to lie in speedy and uniform scoring methods.[6] A department which expects large numbers of applicants for terminal credit might therefore elect to use a USAFI subject examination alone if it believes that *in fact* the examination measures the objectives which the department has established for the course in question. In the contrary case the department might choose to supplement or indeed to replace the USAFI examination by an examination of its own construction. For example, a department whose "general mathematics course" has, among others, the objectives of giving the student an acquaintance with the history of mathematics, an appreciation of mathematics as a fruitful example of logic, and an understanding of the significant place which mathematics occupies in our cultural and intellectual life, would certainly not grant a veteran credit for this course on the basis of USAFI examination results alone.

Whatever device may be adopted for evaluating terminal credit it is important that the "rate of forgetting" of unused concepts and skills be recognized and compensated for. Instructors in the colleges and in the Armed Forces have been startled, for example, by the trigonometric incompetence in the year of 1942, say, of a college man who "earned" an *A* or *B* grade in Trigonometry in a good college in the year of 1939. Since there is every reason for treating the veteran fairly and since there is no conceivable reason for discriminating against him, it is suggested that any individual departments which provide no pre-examination review adjust their terminal credit evaluation standards for time lag. Therefore, a veteran who studied Trigonometry in 1942 by correspondence and who offers this experience for terminal credit in 1945 would be expected to know no more about trigonometry than one of "our own" students who has been away from trigonometry for three years.

b. *Pre-requisite Credit.* The USAFI subject examinations are comprehensive in their coverage of skills and concepts. If a department offers beginning college mathematics in the traditional fashion (algebra, trigonometry, analytical geometry, calculus), a satisfactory total score on a USAFI subject test, coupled with satisfactory scores on each of the subtests, should indicate the veteran's ability to enter the next course in the sequence. Consequently, credit should be

[6] The possibility of using the results of these tests for standardization between departments of mathematics poses an educational question which should be considered independently of the veteran problem. See McGrath, E. J., "The procrustean bed of higher education," School and Society, vol. 61, 1945, pp. 81–84.

given for the pre-requisite course. Here "satisfactory" must be defined by the individual department on the basis of its experience with the test and on the basis of its own standards.

The department which teaches "unified mathematics" on the freshman and/or sophomore levels may find the USAFI sub-test scores of value if it decides that the granting of partial credit in these courses is practicable.

c. *Credit for Majors.* There are men in the Armed Forces and in war industry today who are high in quantitative intelligence, who are interested in mathematics, and who would be concentrating in mathematics on the undergraduate or graduate level had not the war intervened. Some of these men have been able to continue their study of mathematics and other subjects not only through the in-service opportunities mentioned above, but through correspondence with former teachers, independent reading, conversation, and travel. In view of the increasingly critical shortages of professional mathematicians in education, industry, and government it is imperative that educational institutions adopt the most enlightened means of discovering these men and of discerning their present level of mathematical maturity and general educational competence.

For this purpose a department or institution might wish to include some or all of the following measures: (1) A scholastic aptitude test, (2) USAFI college level tests of General Educational Development (Correctness and Effectiveness of Expression, Interpretation of Reading Material in the Social Studies, Interpretation of Reading Material in the Natural Sciences, Interpretation of Literary Materials), (3) USAFI subject tests in mathematics and in any other subjects in which the veteran believes himself to be competent, (4) The Graduate Record Examination[7] profile tests and advanced mathematics test, (5) Short written papers, (6) An informal oral examination, (7) An appraisal of the candidate's formal educational record.

If such a survey reveals that the veteran is relatively mature mathematically and intellectually, and that his formal credits do not measure up to his achievement, it is urged that he be placed immediately in the advanced courses of which he is capable and that appropriate pre-requisite credit be granted.

3. Disseminating Educational and Vocational Advice. Acting under Public Law 346, Servicemen's Readjustment Act of 1944, and under Public Law 16, the Vocational Rehabilitation Program, the Veterans Administration released on March 5, 1945,[8] the names of the first fifty colleges and universities in which Veterans Guidance Centers have been established. All disabled veterans who desire vocational rehabilitation under Public Law 16 will report to these centers for testing and educational and vocational counseling. Veterans who undertake educational courses under Public Law 346 are urged, though not required, to

[7] Published by the Carnegie foundation for the Advancement of Teaching. The word "Graduate" should not imply that these tests are useful only for graduate students. Norms are available for students completing their sophomore year in college. Indeed the writer found the GRE results to be a valuable aid in deciding which fourth class (freshman) cadets at the U. S. Coast Guard Academy were best qualified for an "honors program" in mathematics.

[8] See OWI 4127, Office of War Information, Washington 25, D. C.

take advantage of the services of these centers. Do these Veterans Guidance Centers have adequate information regarding the vocational possibilities of the study of mathematics? The "Dictionary of Occupational Titles,"[9] usually an important aid to counselors, is certainly not too revealing from the vocational standpoint when it states "Mathematician (education). A term applied to a worker who has attained eminence in some field of mathematics and is a recognized authority in that field."

Numerous as the veterans will be in educational institutions in the next few years, their number is relatively small compared with the thirteen or more million civilian students who will undoubtedly enter college in the next ten years. Therefore another important question is: Do the hundreds of educational and vocational counselors in high schools and colleges of the country have adequate information regarding the vocational possibilities of the study of mathematics?

The answer to the above questions must be "NO!" Even though there are some publications which provide important information on the problem[10] it nevertheless appears to be true that in no set of easily accessible documents can the counselor or the student obtain answers to such important questions as: What special abilities and interests are required for success in mathematics; how can these abilities be measured? What special education and training are required; where is it obtainable; how much will it cost? For what vocational areas does the study of mathematics prepare one? What are the employment trends in these vocational areas? What are the beginning and average salaries? What are the associated satisfactions? What prestige is associated with these jobs? What is the principal activity employed in these jobs? Is a preliminary job tryout available? What college courses should supplement mathematics if one of these vocational areas is chosen?

It is gratifying to learn[11] that the Board of Directors of the National Council of Teachers of Mathematics has agreed that the next job of its Commission on Post-War Plans "is to create, or cause to be written, a relatively small but effective pamphlet on mathematics to be used in the guidance of the junior and senior high school pupil." Such a project deserves the full cooperation of all who are interested in mathematics. Since this full cooperation will doubtless be forthcoming, the pamphlet will be a great aid to counselors on the secondary level and it will furnish valuable clues to counselors on the college level. Unfortunately, the completion of such a task is time consuming and it may be a year before the pamphlet is published. The problem of advising the veteran is here and now. The newspapers recently announced that the War Department expects to discharge 1,100,000 in October, 1945, alone. Perhaps as many as 100,000

[9] Part 1, Definition of Titles, U. S. Government Printing Office, 1939.

[10] For example, see Thornton C. Fry's discussion of Industrial Mathematics, reprinted from Research—A National Resource—II, Section VI, Part 4, pp. 268–288, A House Document, 77th Congress. Printed also by the Bell Telephone Laboratories.

Mathematics—Its vocational aspects, Bulletin, Oklahoma A. and M. College, vol. 42, No. 13, May, 1945.

[11] See Zant, J. H., The next step in planning for post-war mathematics, The Mathematics Teacher, October 1945, pp. 276–277.

of these men and women will desire further schooling. How can those veterans who are interested in mathematics learn about mathematics as a career?

Many colleges are establishing centers to which servicemen can come for advice. Many cities are establishing guidance centers for veterans. The Veterans Administration has a "contact man" and a "training officer" near you. All of these agencies are willing and able to place information at the disposal of the veteran. What information can we give these agencies?

If your college has an efficient central placement bureau you can quickly learn where each of your majors, for the past ten years say, found his first job and at what rate of pay. It will be simple to check his advancement and learn the contribution which his training allowed him to make to the war effort. Don't neglect the teacher training placement officer. His records will show both the frequency of calls for high school teachers of mathematics and the average starting pay in comparison with other fields. If no placement bureau records are available, an informal departmental meeting will yield an amazing amount of information about the jobs which your majors have pursued successfully.

All of the information assembled in this fashion should be concisely summarized and appended to information concerning those opportunities for college students of mathematics which are well known to mathematicians but not necessarily known to vocational counselors who are concerned with 25,000 or more occupations. I refer here to college teaching, actuarial work, statistical work, industrial mathematics, the service Academies, and Civil Service positions. This information should then be delivered in person to all agencies associated with veterans in your locality and, of course, any unusual items should be submitted to the Department of General Information in this MONTHLY.

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Summer Meeting, Ithaca, New York, August 19–20, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April 13, 1946
ILLINOIS, Normal, May 10–11, 1946
INDIANA
IOWA, Grinnell, April 19–20, 1946
KANSAS, Emporia, April 6, 1946
KENTUCKY, Louisville, April 27, 1946
LOUISIANA-MISSISSIPPI, Ruston, La., March 15–16, 1946
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, N. Y., May 11, 1946
MICHIGAN, Ann Arbor, April 13, 1946
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, April 4, 1946
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN, Boulder, Colo., April 19–20, 1946
SOUTHEASTERN, Raleigh, N. C., April 19–20, 1946
SOUTHERN CALIFORNIA, Pasadena, March 9, 1946
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WISCONSIN, Milwaukee, May, 1946

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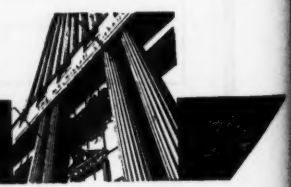

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